Duality Theorems: From Schur-Weyl to Howe

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1 Introduction

> The theory of invariants came into existence about the middle of the nineteenth century somewhat like Minerva: a grown-up virgin, mailed in the shining armor of algebra, she sprang forth from Cayley's Jovian head.
>
> Weyl (1939b, p.489) [26]

Classical invariant theory is concerned with understanding invariants of an algebraic form under the action of a standard classical group, or in other words, understanding the isotypic decomposition of the full tensor algebra for such an action. Both these questions happen to be equivalent to an apparently simpler one, namely the description of the invariants in the full tensor algebra of such an action. This subject, developed in the late 19th century, has since become foundational in modern mathematics, influencing the study of commutative algebra, moduli spaces, and the representation theory of semisimple Lie groups.

One of the first major results in invariant theory was a theorem on the symmetric functions that described the invariants of the symmetric group by permutations of the variables. This is often called the Schur-Weyl Duality, which will be the first topic this exposition deals with. Invariant theory of infinite groups is inextricably linked with the development of linear algebra, especially the theories of quadratic forms and determinants. An extension of the Schur-Weyl Duality for infinite groups will be dealt with for a certain class of representations – locally regular representations – and will be the topic of interest for the second part of this report.

A pivotal moment in the development of invariant theory was David Hilbert's 1890 [12] proof of the finite generation of invariant algebras, which played a crucial role in the emergence of abstract algebra. His subsequent 1893 work [13] approached the problem from a more geometric perspective, though it remained relatively obscure until David Mumford revived and significantly generalized these ideas in the 1960s through geometric invariant theory. His ideas developed the modern framework of the study of algebraic group actions on affine and projective varieties, linking invariant theory with contemporary algebraic geometry.

To illustrate the core ideas of classical invariant theory, consider a vector space V and a group G — either a finite group or a classical Lie group — acting linearly on V. This induces a corresponding action on the polynomial function space $R(V)$ on V by the formula $g \cdot f(v) = f(g^{-1}v)$, $g \in G, v \in V$. The subalgebra of invariant functions under the "change of variable" due to the action of G consist of those f satisfying $g \cdot f = f$ for all $g \in G$. They form a commutative algebra denoted $A = R(V)^G$ and this algebra is interpreted as the algebra of functions on the "invariant theory quotient" $V/\!\!/G$, because any one of these functions gives the same value for all points that are equivalent (that is, $f(v) = f(gv)$ for all g). In the language of modern algebraic geometry, $V/\!\!/G = \operatorname{Spec} A = \operatorname{Spec} R(V)^G$.

A fundamental question in invariant theory is whether the algebra A is finitely generated. Hilbert's proof [12] mentioned earlier established finite generation when G is the general linear group, ensuring that $V/\!\!/G$ remains an affine algebraic variety. However, the broader question — whether finite generation holds for arbitrary groups G — was posed as Hilbert's 14th problem. Nagata later demonstrated that the answer is negative in general [23], providing a counterexample.

Despite this, advances in representation theory during the early 20th century identified a large class of groups, known as reductive groups, for which finite generation does hold. These include all finite groups and classical matrix groups. The final section of this exposition will explore reductive groups, particularly in relation to Howe duality. Howe Duality has strong ties with classical invariant theory, as discussed in [14]. The Howe Duality also yields the local theta correspondence, which relates irreducible admissible representations over a local field. There also exists a global theta correspondence, which relates irreducible automorphic representations over a global field, that one can read more about in [7] or [24, §8.] for example.

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2 Schur-Weyl Duality

In this section, we describe the irreducible representations of the general linear group $\mathrm{GL}_n(\mathbb{C})$ and the symmetric group of d elements S_d. We proceed to demonstrate a correspondence between the parametrisation of isomorphism classes of irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ on $(\mathbb{C}^n)^{\otimes d}$ with those of S_d on $(\mathbb{C}^n)^{\otimes d}$.

2.1 Irreducible representations of S_d and $\mathrm{GL}_n(\mathbb{C})$

Recall that the symmetric group of d permutations S_d has exactly as many irreducible representations as it does conjugacy classes, and each conjugacy class is a cycle-type equivalence class. There is a one-to-one canonical correspondence between the set of cycle-types and the ways to write n as the sum of positive integers. A fundamental tool for studying the representations of S_d via this canonical bijection with partitions is the Young Diagram, which is an array of boxes: for a given partition $d = \lambda_1 + \lambda_2 + \ldots + \lambda_k$ (which we denote by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$) where $\lambda_i \geq \lambda_{i+1}$ for every $1 \leq i \leq k$, we draw a row of λ_1 boxes. Beneath it, we draw a row of λ_2 boxes, and so on so forth so that the last row contains λ_k boxes and each row is as long or shorter than the one above it. For more details, see [6, Chapter 4].

The irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ are slightly less obvious. We will give a brief overview without fleshing out any details and refer the reader to more comprehensive texts on the same topic, see [6, Chapter 15, §15.5] for complete details or [4, Appendix A] for a quick overview. First, we want to consider finite dimensional representations (π, V) of $G = \mathrm{GL}_n(\mathbb{C})$, where $\pi : G \to \mathrm{GL}(V)$ is a *regular (or algebraic)* map. This means that we identify $\mathrm{GL}(V)$ with $\mathrm{GL}_m(\mathbb{C})$ (for some m) and let $g = (g_{ij}) \in G$. Then $\pi(g) = (\pi(g)_{kl})$ is a matrix. The regularity of (π, V) means that the matrix coefficients $\pi(g)_{kl}$ are polynomials in g_{ij} and in $\det(g)^{-1}$. If $\det(g)^{-1}$ does not appear, it is called a *polynomial* representation. Then, one can use the theory of Lie algebras to derive the irreducible representations of $\mathrm{GL}_n(\mathbb{C})$.

Let T be the subgroup of $G = \mathrm{GL}_n(\mathbb{C})$ consisting of diagonal matrices. A regular character of T is called a *weight*. The group Λ of weights is called the *weight lattice*, which may be identified with $\mathbb{Z}^n$ as follows. If $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{Z}^n$ [1], we use the notation

$$t \mapsto t^\lambda, \quad t = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}, \quad t^\lambda = \prod_{i=1}^n t_i^{\lambda_i}.$$

Let (π, V) be a representation of $\mathrm{GL}_n(\mathbb{C})$. If λ is a weight, then

$$V_\lambda = \{v \in V \mid \pi(t)v = t^\lambda v, \quad t \in T\}$$

is called the *weight space* of λ. If $V_\lambda \neq 0$, then λ is called the *weight* of the representation π. A weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ is called *dominant* if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. One can define a partial order on these set of weights, denoted by say, $\succcurlyeq$. A weight is *maximal* for a representation π if λ is a weight that is maximal with respect to $\succcurlyeq$. If π is irreducible, we use *highest weight* if λ is a maximal weight. Then, one can show that there is a bijection between the set of isomorphism classes of irreducible representations of $\mathrm{GL}_n(\mathbb{C})$ and highest weights ([6, Proposition 15.47]).

One can go a step further by using Weyl's unitary trick to show that every regular representation of $\mathrm{GL}_n(\mathbb{C})$ is determined by its restriction to the unitary group $U(n)$, whose continuous representations are classified by their characters, that are symmetric Laurent polynomials in n variables. Thus, algebraic representations of $\mathrm{GL}_n(\mathbb{C})$ are also determined by such characters. Then, by the Peter–Weyl Theorem and Cauchy's identity, irreducible *polynomial* representations of $\mathrm{GL}_n(\mathbb{C})$ correspond to Schur polynomials—symmetric polynomials in n variables indexed by partitions.[2] For $\pi = \pi_\lambda^{\mathrm{GL}_n(\mathbb{C})}$, the irreducible representation with

[1] This notation is suggestive... deja vu? However, note that here we allow negative integers for now.

[2] Finite dimensional representations of $\mathrm{GL}_n(\mathbb{C})$, both as an algebraic group or as a complex Lie group are indexed by n-tuples $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. Such a representation is polynomial if and only if $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq 0$, in which case people often think of this as a partition.

highest weight λ, the Weyl character formula tells us that $\chi_\pi(g) := \mathrm{tr}(\pi(g))$ is a homogeneous polynomial $s_\lambda = s_\lambda^{(n)}$, which can be written as $s_\lambda(t_1, \ldots, t_n) = \det(t^{\lambda_j+n-1-j})/\det(t_i^{n-1-j})$. These s_λ is are what we called Schur polynomials (see [4, Appendix A.1] for more details).

We end this brief overview with the following Proposition:

Proposition 2.1. *Let λ be any dominant weight for $\mathrm{GL}_n(\mathbb{C})$ that is a partition of $d \leq n$. Then, $\pi_\lambda^{\mathrm{GL}_n(\mathbb{C})}$ appears in the $\mathrm{GL}_n(\mathbb{C})$-module $(\mathbb{C}^n)^{\otimes d}$.*

Proof. See [4, Proposition A.7]. □

Essentially, the set of isomorphism classes of irreducible polynomial representations of $\mathrm{GL}_n(\mathbb{C})$ can be indexed by partitions. Given that both S_d and $\mathrm{GL}_n(\mathbb{C})$ have their irreducible representations indexed by partitions of n (or d) might suggest to the reader that these two objects could speak to each other in some sense.

2.2 Schur-Weyl Duality and Explicit Examples

The Schur-Weyl Duality (sometimes referred to as Frobenius-Schur Duality or Schur Duality) is a relationship between the representation theory of $G = \mathrm{GL}_n(\mathbb{C})$ and S_d. Consider the defining module $\mathbb{C}^n$ of G. Both G and S_d act naturally on $(\mathbb{C}^n)^{\otimes d}$. G acts diagonally via

$$g \cdot (v_1 \otimes \ldots \otimes v_d) = g \cdot v_1 \otimes g \cdot v_2 \otimes \ldots \otimes g \cdot v_d.$$

S_d acts by permuting the entries (we write this as a right action) as

$$(v_1 \otimes \ldots \otimes v_d) \cdot w = v_{w(1)} \otimes v_{w(2)} \otimes \ldots \otimes v_{w(d)}.$$

So, we may view $(\mathbb{C}^n)^{\otimes d}$ as a $(G \times S_d)$-module.

Theorem 2.2 (Schur-Weyl Duality)**.** *The $\mathrm{GL}_n(\mathbb{C}) \times S_d$-module $(\mathbb{C}^n)^{\otimes d}$ decomposes as follows:*

$$(\mathbb{C}^n)^{\otimes d} = \bigoplus_\lambda \pi_\lambda^{\mathrm{GL}_n(\mathbb{C})} \otimes \pi_\lambda^{S_d},$$

where λ runs through the partitions of d of length $\leq n$; $\pi_\lambda^{\mathrm{GL}_n(\mathbb{C})}$ is the irreducible representation of $\mathrm{GL}_n(\mathbb{C})$ of highest weight λ. $\pi_\lambda^{S_d}$ is the irreducible representation of S_d corresponding to the partition λ. Each irreducible representation of $\mathrm{GL}_n(\mathbb{C})$ whose highest weight is a partition of d appears in this decomposition. Moreover, if $n \geq d$, then every irreducible representation of S_d occurs in this decomposition exactly once.

The proof for Schur-Weyl Duality involves careful combinatorial calculations, so we postpone the proof for later, where we prove it systematically and in more generality (c.f. Theorem 3.17).

This theorem yields not only the fact that the representation of $\mathrm{GL}_n(\mathbb{C}) \times S_d$ breaks down into distinct decomposable bits, but also the fact there is a correspondence within each bit $\pi_\lambda^{S_d} \longleftrightarrow \pi_\lambda^{\mathrm{GL}_n(\mathbb{C})}$. This notion of correspondence has been emphasized by Howe, as we will see in section 4.

Why is this correspondence special? If A and B are arbitrary groups, then knowing how a representation V decomposes as a representation of A, and separately how it decomposes as a representation of B, does not in general determine its decomposition as a representation of $A \times B$. This is because such information does not specify how irreducibles of A pair with irreducibles of B.

A simple counterexample occurs if we consider $A = B = C_2$, the cyclic group of order 2, acting on a two-dimensional vector space V. Suppose that, as a representation of either A or B, V decomposes as a direct sum of the trivial representation 1 and the sign representation -1. In this case, V could be isomorphic to either

$$(1 \otimes 1) \oplus ((-1) \otimes (-1)) \quad \text{or} \quad (1 \otimes (-1)) \oplus ((-1) \otimes 1),$$

and the separate decompositions do not determine which pairing occurs. More generally, such ambiguity arises whenever A and B both admit non-isomorphic irreducible representations of the same dimension.

To overcome this, one can proceed as follows. If the action of A is understood, then V admits a canonical decomposition of the form

$$V \cong \bigoplus_i V_i \otimes \mathrm{Hom}_A(V_i, V),$$

where the V_i are irreducible subrepresentations of A, and $\mathrm{Hom}_A(V_i, V)$ is the multiplicity space of V_i in V. Writing V simply as a direct sum of several copies of the V_i loses this canonical structure. We will see in the next section that the advantage of this tensor product decomposition is that if the action of B commutes with the action of A, then B naturally acts on each multiplicity space $\mathrm{Hom}_A(V_i, V)$, through which one can recover the structure of V as a representation of $A \times B$.

We now compute a few explicit examples, illustrating the result of Theorem 2.2. In order to do so, we introduce an element of the group algebra $\mathbb{C}[S_d]$, called the Young symmetriser c_λ, following [11, Chapter 9]. The Young symmetriser has a natural action on tensor products $(\mathbb{C}^n)^{\otimes d}$ and has as image an irreducible representation of $\mathrm{GL}_n(\mathbb{C})$, in other words, the Young symmetriser is a projection operator onto a $\mathrm{GL}_n(\mathbb{C})$-irreducible subspace of $(\mathbb{C}^n)^{\otimes d}$. For a Young diagram, number the boxes, say consecutively as shown[3]:

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2} 6 & 7 \\ \cline{1-2} 8 \\ \cline{1-1}\end{array}$$

Define two permutation subgroups as follows:

$$P_\lambda = \{g \in S_d : g \text{ preserves each row of } \lambda\} \quad \text{and} \quad Q_\lambda = \{g \in S_d : g \text{ preserves each column of } \lambda\}.$$

So, in our above example, we would have elements in $P_\lambda = \{e, (123), (213), (45), (67)\}$ and elements in $Q_\lambda = \{e, (1468), (1486), (1864), (6418), \ldots, (257), \ldots\}$. Corresponding to these two subgroups, define two vectors in the group algebra $\mathbb{C}[S_d]$ as

$$a_\lambda = \sum_{g \in P_\lambda} e_g, \qquad b_\lambda = \sum_{g \in Q_\lambda} \mathrm{sgn}(g) e_g$$

where e_g is the unit vector corresponding to g and $\mathrm{sgn}(g)$ is the sign of the permutation. The product $c_\lambda = a_\lambda b_\lambda$ is called the Young symmetriser corresponding to λ.

Example 2.3. Let $d = 2$ and n be integer greater than 1. Then, the partitions of d are given by $\lambda_{(1,1)} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$ and $\lambda_{(2)} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$. Let $v_1, v_2 \in \mathbb{C}^n$. So, $c_{\lambda_1} := c_{(1,1)} = e_1(e_1 - e_{(1\ 2)}) = e_1 - e_{(1\ 2)}$. Then,

$$(v_1 \otimes v_1) \cdot c_{\lambda_1} = v_1 \otimes v_1 - v_1 \otimes v_1 = 0$$
$$(v_1 \otimes v_2) \cdot c_{\lambda_1} = v_1 \otimes v_2 - v_2 \otimes v_1$$

So, the image of c_{λ_1} is given by the exterior product $\bigwedge^2 \mathbb{C}^n$. Similarly, for $c_{\lambda_2} = c_{(2)}$, we have $c_{\lambda_2} = e_1 + e_{(1\ 2)}$, and we obtain the image of c_{λ_2} to be $\mathrm{Sym}^2\mathbb{C}^n$. Thus, $\mathbb{C}^n \otimes \mathbb{C}^n = \mathrm{Sym}^2\mathbb{C}^n \oplus \bigwedge^2 \mathbb{C}^n$. Thus, the space of two tensors decomposes into two parts – symmetric and anti-symmetric parts, each of which are irreducible modules of $\mathrm{GL}_n(\mathbb{C})$. So, there is a duality between the trivial and signature representations of S_2 corresponding to the symmetric square and alternating square representations of $\mathrm{GL}_n(\mathbb{C})$. Thus, we have $(\mathbb{C}^n)^{\otimes 2} = \left(\mathrm{Sym}^2\mathbb{C}^n \otimes \mathrm{triv}\right) \oplus \left(\bigwedge^2 \mathbb{C}^n \otimes \mathrm{sgn}\right)$.

[3]This is just a canonical numbering – one can number a tableau in any order with integers $1, \ldots, d$. If an ordering apart from the canonical one was chosen, one would get different elements in place of P_λ and Q_λ, but the representations constructed this way would be isomorphic.

Example 2.4. Let $d = 3$ and $n \geq 2$. Then, the partitions are given by $\lambda_{(1,1,1)} = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array}$, $\lambda_{(2,1)} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$,
and $\lambda_{(3)} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array}$.

Then, similar to Example 2.3, we obtain that the image of $c_{(1,1,1)}$ in $\mathbb{C}S_d$ yields $\bigwedge^3 \mathbb{C}^n$ and $c_{(3)}$ yields $\mathrm{Sym}^3\mathbb{C}^n$. For $c_{(2,1)}$, we compute $c_{(2,1)} = (e_1 + e_{(1\ 2)})(e_1 - e_{(1\ 3)}) = e_1 - e_{(1\ 3)} + e_{(1\ 2)} - e_{(1\ 3\ 2)}$. Then,

$$
\begin{aligned}
c_{(2,1)}(v_1 \otimes v_1 \otimes v_1) &= v_1 \otimes v_1 \otimes v_1 - v_1 \otimes v_1 \otimes v_1 + v_1 \otimes v_1 \otimes v_1 - v_1 \otimes v_1 \otimes v_1 = 0 \\
c_{(2,1)}(v_1 \otimes v_2 \otimes v_3) &= v_1 \otimes v_2 \otimes v_3 - v_3 \otimes v_2 \otimes v_1 + v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_1 \otimes v_2 \\
c_{(2,1)}(v_1 \otimes v_1 \otimes v_2) &= 2\,(v_1 \otimes v_1 \otimes v_2) - 2\,(v_2 \otimes v_1 \otimes v_1) \\
c_{(2,1)}(v_1 \otimes v_2 \otimes v_1) &= v_2 \otimes v_1 \otimes v_1 - v_1 \otimes v_1 \otimes v_2 \\
c_{(2,1)}(v_2 \otimes v_1 \otimes v_1) &= v_2 \otimes v_1 \otimes v_1 - v_1 \otimes v_2 \otimes v_2
\end{aligned}
$$

Let us denote the space spanned by the image of $c_{(2,1)}$ as $V_{(2,1)} = \langle v_1 \otimes v_2 \otimes v_3 - v_3 \otimes v_2 \otimes v_1 + v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_1 \otimes v_2, v_2 \otimes v_1 \otimes v_1 - v_1 \otimes v_2 \otimes v_2\rangle$. Furthermore, we know that the irreducible representations of S_3 are the one-dimensional trivial representation, one-dimensional sign representation and the two-dimensional standard representation. Thus, $(\mathbb{C}^n)^{\otimes 3} \cong \left(\mathrm{Sym}^3\mathbb{C}^n \otimes \mathrm{triv}\right) \oplus \left(V_{(2,1)} \otimes \mathrm{std}\right) \oplus \left(\bigwedge^3 \mathbb{C}^n \otimes \mathrm{sgn}\right)$.

3 General Duality Theorem

The Schur-Weyl Duality gave us a correspondence between the irreducible representations of S_d and irreducible polynomial representations of $\mathrm{GL}_n(\mathbb{C})$. It is however not true that we can obtain such a correspondence for all representations of $\mathrm{GL}_n(\mathbb{C})$, or any arbitrary group. So, if we wish to generalise this duality, we need to work with specific types of groups and representations. We largely follow the exposition in [11].

3.1 Reductive groups

Definition 3.1. A subgroup $G \subset \mathrm{GL}_n(\mathbb{C})$ is a *linear algebraic group* is there if a set of polynomial functions $\mathcal{A}$ on $M_n(\mathbb{C})$ such that $G = \{g \in \mathrm{GL}_n(\mathbb{C}) : f(g) = 0 \text{ for all } f \in \mathcal{A}\}$. For example

(i) $\mathcal{A} = \det(g) - 1 = 0$, then $G = \mathrm{SL}_n(\mathbb{C})$.
(ii) $\mathcal{A} = x_{ij} = 0$ for every $i \neq j$ then $G = T$, the set of diagonal matrices.
(iii) $x_{ij} = 0$ for all $i > j$, then $G = B$ the Borel subgroup consisting of upper triangular matrices.

As mentioned in §2.1, for $G = \mathrm{GL}_n(\mathbb{C})$, the algebra of *regular functions* is defined as $\mathcal{O}[\mathrm{GL}_n(\mathbb{C})] = \mathbb{C}[x_{11}, x_{12}, \ldots, x_{nn}, \det(x)^{-1}]$. For $\mathrm{GL}(V)$, we let $\varphi : \mathrm{GL}(V) \to \mathrm{GL}_n(\mathbb{C})$ be a group isomorphism defined in terms of a basis for V. The algebra $\mathcal{O}[\mathrm{GL}(V)]$ of regular functions is all functions $f \circ \varphi$, where f is regular on $\mathrm{GL}_n(\mathbb{C})$. Similarly, if $G \subset \mathrm{GL}(V)$ is an algebraic subgroup, then a complex valued function f on G is regular if it is the restriction to G of a regular function on $\mathrm{GL}(V)$.

We can also define regular maps – let G and H be linear algebraic groups and let $\varphi : G \to H$ be a map. For $f \in \mathcal{O}[H]$, define the function $\varphi^*(f)$ on G by $\varphi^*(f)(g) = f(\varphi(g))$. We say that φ is a *regular map* if $\varphi^*(\mathcal{O}[H]) \subseteq \mathcal{O}[G]$.

Surprise, we can also define regular representations; but before we get there, we need to specify what we mean by an algebraic group homomorphism. A map $\varphi : G \to H$ is an *algebraic group homomorphism* if it is a group homomorphism that is also a regular map. A representation of a linear algebraic group G is a pair (ρ, V) that is an algebraic group homomorphism.

Definition 3.2. If the dimension of V is finite, a representation is *regular* if the functions on G, $G \to \mathbb{C}$; $g \mapsto \langle v^*, \rho(g)v\rangle$, called *matrix coefficients*, are regular for every $v \in V$ and $v^* \in V^*$ (here $\langle v^*, v\rangle := v^*(v)$).

Regular representations are also called *rational representations* since each matrix coefficient is a rational function of the matrix entries of g.[4]

It so happens that these rational representations for most groups one encounters (think finite groups and classical groups), have a really nice property – namely that of complete reducibility. A rational representation (ρ, V) of a linear algebraic group G is *completely reducible* if for every G-invariant subspace $W \subset V$, there exists a G invariant subspace $U \subset V$ such that $V = W \oplus U$. To reflect this property, we have the following definition:

Definition 3.3. A linear algebraic group G is a *reductive group* if every rational representation (ρ, V) of G is completely reducible.

One can think of reductiveness as the analogue of complete reducibility of compact Lie groups. The proof of the fact that classical groups are reductive can be found in [11, §3.3.1].

But, what if the representation V of any linear algebraic group is infinite dimensional? Not all hope is lost, for then we have the notion of locally regular representations.

Definition 3.4. If dimension of V is infinite, then (ρ, V) is a *locally regular representation* if every finite dimensional subspace $E \subset V$ is contained in a finite-dimensional G-invariant subspace F such that the restriction of ρ to F is regular.

3.2 Representation theory of linear algebraic groups

In this section, we record some fundamental results in the representation theory of linear algebraic groups. Before we begin, we generalise the set up to work over associative algebras over $\mathbb{C}$.

Definition 3.5. An *associative algebra* over the complex field $\mathbb{C}$ is a vector space $\mathcal{A}$ over $\mathbb{C}$ together with a bilinear multiplication map

$$\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}, \qquad x, y \mapsto xy = \mu(x, y)$$

such that $(xy)z = x(yz)$. We will always assume our algebra has an identity element.

Definition 3.6. A *representation*, or *module*, of $\mathcal{A}$ is a pair (ρ, V) where $\rho : \mathcal{A} \to \mathrm{End}(V)$ is an algebra homomorphism.

When the map ρ is understood from context, we shall call V an $\mathcal{A}$-module. We say a vector space V has *countable dimension* if the cardinality of every linearly independent set of V is countable. Schur's lemma asserts the only $\mathcal{A}$-module homomorphisms between two irreducible (possibly infinite-dimensional but with countable dimension) representations are multiplication by scalars (c.f. [11, Lemma 4.1.4]).

Definition 3.7. Let (ρ, V) and (τ, W) be two representations of $\mathcal{A}$ and let $\mathrm{Hom}(V, W)$ be the space of $\mathbb{C}$-linear maps from V to W. We denote by $\mathrm{Hom}_{\mathcal{A}}(V, W)$ to be the set of all $T \in \mathrm{Hom}(V, W)$ such that $T\rho(a) = \tau(a)T$ for all $a \in \mathcal{A}$. Such a map is called an *intertwining operator* between two representations or a *module homomorphism.*

Definition 3.8. An $\mathcal{A}$-module (possibly infinite dimensional) is *irreducible* if the only submodules are $\{0\}$ and $\mathcal{A}$.

Definition 3.9. A finite-dimensional $\mathcal{A}$-module V is *completely reducible* if for every $\mathcal{A}$-invariant subspace $W \subset V$, there exists a complementary $\mathcal{A}$-invariant subspace U such that $V = W \oplus U$.

We state a proposition that gives some equivalent characterisations of complete reducibility:

Proposition 3.10. *For a finite-dimensional representation (ρ, V), the following are equivalent:*

[4] If you're wondering if the name suggests something analogous to the regular representation you studied about in your first course on representation theory, then you are correct. Every finite-dimensional representation of a linear algebraic group G arises as a subrepresentation of a direct sum of copies of the regular representation (Corollary 4.13 [19]).

(i) (ρ, V) *is completely reducible.*
(ii) $V + W_1 \oplus \ldots \oplus W_m$ *with each* W_i *an irreducible* $\mathcal{A}$*-module.*
(iii) $V = V_1 + \ldots + V_d$ *as a vector space, where each* V_i *is an irreducible* $\mathcal{A}$*-submodule.*
Furthermore, if all the V_i *in (iii) are equivalent to a single irreducible* $\mathcal{A}$*-module* W*, then every* $\mathcal{A}$*-submodule of* V *is isomorphic to a direct sum of copies of* W*.*

We can also extend this notion of complete reducibility to infinite dimensional vector spaces.

Definition 3.11. An $\mathcal{A}$-module is *locally completely reducible* if for every $v \in V$, the cyclic submodule $\mathcal{A} \cdot v$ is finite-dimensional and completely reducible.

Now, we wish to characterise the decomposition of these locally completely reducible representations. Let $\widehat{\mathcal{A}}$ denote the set of all equivalence classes of finite-dimensional irreducible $\mathcal{A}$-modules. Suppose V is an $\mathcal{A}$-module. For each $\lambda \in \widehat{\mathcal{A}}$, define the λ*-isotypic subspace* to be

$$V_{(\lambda)} = \sum_{U \subset V, [U] = \lambda} U,$$

where $[U]$ denotes the equivalence class of all $\mathcal{A}$-submodules equal to U. Let F^λ denote a representative for each $\lambda \in \widehat{\mathcal{A}}$. Then, there is a tautological map

$$S_\lambda : \mathrm{Hom}_{\mathcal{A}}(F^\lambda, V) \otimes F^\lambda \to V, \quad S_\lambda(u \otimes v) = u(v).$$

We can make $\mathrm{Hom}_{\mathcal{A}}(F^\lambda, V)$ an $\mathcal{A}$-module by defining $x \cdot (u \otimes v) = u \otimes (x \cdot v)$. Then, S_λ is an intertwining map.

Proposition 3.12. *([11, Proposition 4.1.15]) Let* V *be a locally completely reducible* $\mathcal{A}$*-module. Then,* S_λ *is an* $\mathcal{A}$*-module isomorphism onto* $V_{(\lambda)}$*. Furthermore,*

$$V \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}} V_{(\lambda)} \cong \bigoplus_{\lambda \in \widehat{\mathcal{A}}} \mathrm{Hom}_{\mathcal{A}}(F^\lambda, V) \otimes F^\lambda.$$

This proposition is what enables us to decompose our representation when we discuss the general duality theorem for group representations in §3.3.

Proof. If $U \subseteq V$ is an $\mathcal{A}$-invariant finite-dimensional irreducible subspace with $[U] = \lambda$, then there exists $u \in \mathrm{Hom}_{\mathcal{A}}(F^\lambda, V)$ such that $\mathrm{Range}(u) = U$. Hence S_λ is surjective. To show S_λ is injective, let $u_i \in \mathrm{Hom}_{\mathcal{A}}(F^\lambda, V)$ and $w_i \in F^\lambda$ for $1 \le i \le k$, and suppose that $\sum_i u_i(w_i) = 0$. We may assume that $\{w_1, \ldots w_k\}$ are linearly independent and that $u_i \neq 0$ for all i. Fix an i. Let $W = u_1(F^\lambda) + \ldots + u_k(F^\lambda)$. Then W is a finite-dimensional $\mathcal{A}$-submodule of $V_{(\lambda)}$. Hence by Proposition 3.10, $W = W_1 \oplus \ldots \oplus W_m$ with W_j irreducible and $[W_j] = \lambda$. Let $\varphi_j : W \to F^\lambda$ be the projection onto the subspace W_j. Then $\varphi_j \circ u_i \in \mathrm{End}_{\mathcal{A}}(F^\lambda)$. Thus $\varphi_j \circ u_i = c_{ij} I$ with $c_{ij} \in \mathbb{C}$ (Schur's Lemma), and we have

$$0 = \sum_i \varphi_j u_i(w_i) = \sum_i c_{ij} w_i \quad \text{for } j = 1, \ldots, m.$$

Since $\{w_1, \ldots, w_k\}$ are linearly independent, we conclude that $c_{ij} = 0$. Hence, the projection of $\mathrm{Range}(u_{ij})$ onto W_j is zero for $1 \le j \le m$. This implies that $u_i = 0$, proving injectivity of S_λ.

The definition of local complete reducibility implies that V is spanned by the spaces $V_{(\lambda)}$, $\lambda \in \widehat{\mathcal{A}}$. So, it remains to prove that these spaces are linearly independent. Fix distinct classes $\{\lambda_1, \ldots, \lambda_d\} \subseteq \widehat{\mathcal{A}}$ such that $V_{(\lambda_i)} \neq \{0\}$. We will prove via induction on d that the sum $E = V_{(\lambda_1)} + \ldots + V_{(\lambda_d)}$ is direct. If $d = 1$, there is nothing to prove. Let $d > 1$ and assume that the result holds for $d - 1$ summands. Set $U = V_{(\lambda_1)} + \ldots + V_{(\lambda_{d-1})}$. Then $E = U + V_{(\lambda_d)}$ and $U = V_{(\lambda_1)} \oplus \ldots \oplus V_{(\lambda_{d-1})}$ by the induction hypothesis. For $i < d$, let $p_i : U \to U_{(\lambda_i)}$ be the projection corresponding to this direct sum decomposition. Suppose for the sake of contradiction, that there exists a non-zero vector $v \in U \cap V_{(\lambda_d)}$. The $\mathcal{A}$-submodule $\mathcal{A}v$ of $V_{(\lambda_d)}$. is then non-zero, finite-dimensional, and completely reducible. Hence by the last part of Proposition 3.10, there is a decomposition

$$\mathcal{A}v = W_1 \oplus \ldots \oplus W_r, \quad \text{with} \quad [W_i] = \lambda_d. \tag{1}$$

On the other hand, since $\mathcal{A}v \subset U$, there must exist an $i < d$ such that $p_i(\mathcal{A}v)$ is non-zero. But Proposition 3.10 then implies that $p_i(\mathcal{A}v)$ is a direct sum of irreducible modules of type λ_i. Since $\lambda_i \neq \lambda_d$, this contradicts (1), by Schur's Lemma. hence $U \cap V_{(\lambda_d)} = (0)$, and we have $E = V_{(\lambda_1)} \oplus \ldots \oplus V_{(\lambda_d)}$.

□

We now move forward with other fundamental results in the representation theory of linear algebraic groups, which will be used in the proof of the Duality Theorem.

Theorem 3.13 (Jacobson Density Theorem)**.** *Let V be a countable-dimensional vector space over $\mathbb{C}$. Let $\mathcal{R}$ be a subalgebra of* $\mathrm{End}(V)$ *that acts irreducibly on V. If $v_1, \ldots, v_n$ are linearly independent in V, then, for any $w_1 \ldots, w_n$ in V, there exists $T \in \mathcal{R}$ such that $Tv_i = w_i$ for all $1 \leq i \leq n$.*

Proof. See [11, Theorem 4.1.5].

□

Another important theorem in the theory is the Double Centraliser Theorem.

Definition 3.14. Let V be a vector space. For any subset $\mathcal{S} \subset \mathrm{End}(V)$, we define $\mathrm{Cent}(\mathcal{S}) := \{x \in \mathrm{End}(V) : xs = sx \text{ for all } s \in \mathcal{S}\}$.

Clearly, $\mathrm{Cent}(\mathrm{Cent}(\mathcal{S})) \supseteq \mathcal{S}$, but it is not clear when one can say that the two sets are equal. The Double Centraliser Theorem gives conditions under which this equality occurs.

Theorem 3.15 (Double Centraliser Theorem)**.** *Suppose $\mathcal{A} \subset$* $\mathrm{End}(V)$ *is an associative algebra with identity I_V. Set $\mathcal{B} = Cent(\mathcal{A})$. If V is a completely reducible $\mathcal{A}$-module, then $Cent(\mathcal{B}) = \mathcal{A}$.*

Notation. If V is a complex vector space, $v_j \in V$, and $T \in \mathrm{End}(V)$, then we write

$$V^{(n)} = \underbrace{V \oplus \cdots \oplus V}_{n \text{ times}} \quad \text{and} \quad T^{(n)}[v_1, \ldots, v_n] = [Tv_1, \ldots, Tv_n].$$

The proof of the theorem involves using the following lemma which we state without proof.

Lemma 3.16. *([11, Corollary 4.1.12]) Suppose (ρ, V) and (τ, W) are completely reducible representations of $\mathcal{A}$, then $(\rho \oplus \tau, V \oplus W)$ is a completely reducible representation.*

Proof of Theorem 3.15. We know that $\mathrm{Cent}(\mathcal{B}) \supseteq \mathcal{A}$. Fix a basis $\{v_1, \ldots, v_n\}$ in V. In order to show containment in the other direction, it would suffice to show that for any $T \in \mathrm{Cent}(\mathcal{B})$, there exists an element $S \in \mathcal{A}$ such that $Sv_i = Tv_i$ for all $1 \leq i \leq n$. Let $w_0 = v_1 \oplus \ldots \oplus v_n \in V^{(n)}$. Since $V^{(n)}$ is a completely reducible $\mathcal{A}$-module by Lemma 3.16, the cyclic submodule $M = \mathcal{A} \cdot w_0$ has an $\mathcal{A}$-invariant complement. So, there is a projection $P : V^{(n)} \to M$ that commutes with $\mathcal{A}$. The action of P is given by an $n \times n$ matrix $[p_{ij}]$ where $p_{ij} \in \mathcal{B}$. As $Pw_0 = w_0$ and $Tp_{ij} = p_{ij}T$, we have $P(Tv_1 \oplus \ldots \oplus Tv_n) = Tv_1 \oplus \ldots \oplus Tv_n \in M$. So, by the definition of M which is a cyclic submodule, there exists an $S \in \mathcal{A}$ such that $Sv_1 \oplus \ldots \oplus Sv_n = Tv_1 \oplus \ldots \oplus Tv_n$. This means that $T = S$, and so we are done. □

3.3 Duality Theorem

Let $G \subseteq \mathrm{GL}_n(\mathbb{C})$ be a reductive linear algebraic group. Let $\widehat{G}$ denote the set of equivalence classes of the irreducible regular representations of G, and fix a representation (π^λ, F^λ) for each $\lambda \in \widehat{G}$. Let L be a locally regular representation. We hope to apply Proposition 3.12 to our case. We can define an associative *group algebra* $\mathcal{A}[G]$ of G as follows: as a vector space, $\mathcal{A}[G]$ is the set of all functions $f : G \to \mathbb{C}$ such that support of f is finite. One can define the set of functions $\{\delta_g : g \in G\}$ where $\delta_g(x) = 1$ if $x = g$, and $\delta_g(x) = 0$ otherwise. Thus, every element x of $\mathcal{A}[G]$ has a unique expression as a formal sum $\sum_{g\in G} x(g)\delta_g$. Given $x, y \in \mathcal{A}[G]$, we can define a convolution product $x * y$ by

$$\left(\sum_{g\in G} x(g)\delta_g\right) * \left(\sum_{h\in G} y(h)\delta_h\right) = \sum_{g,h\in G} x(g)y(h)\delta_{gh}.$$

Then, one can show that representations and modules over the group algebra are essentially the same thing.

Given the bijection between representations of G and $\mathcal{A}[G]$-modules, one can see that if G is a reductive group, then a locally regular representation (ρ, L) of G is a locally completely reducible $\mathcal{A}[G]$-module. This is because for any $v \in L$, since v is locally regular, we have $\mathcal{A}[G]\cdot v \subset W$ for some G-invariant subspace $W \subset V$. As $\mathcal{A}[G]\cdot v \subset \mathcal{A}[G]\cdot W \subset W$, the cyclic module $\mathcal{A}[G]\cdot v$ is regular, and in particular finite-dimensional. As G is reductive, $\mathcal{A}[G]\cdot v$ is completely reducible. Further, we need to check that the irreducible $\mathcal{A}[G]$-submodules are of L are irreducible regular representations of G. To see this, let $V \subset L$ be one such G-invariant subspace that is irreducible. We wish to show that $(\rho\mid_V, V)$ is a regular representation of G. Indeed, as L is locally regular, there exists a subspace $W \subset L$ such that $V \subset X$ and $\rho\mid_W$ is regular. After a change of basis, we can write $\rho\mid_V$ as a submatrix of $\rho\mid_W$, so $\rho\mid_V$ must be regular. Thus, by Proposition 3.12, we have as $\mathcal{A}[G]$-modules

$$L \cong \bigoplus_{\lambda\in\mathrm{Spec}(\rho)} \mathrm{Hom}_{\mathcal{A}[G]}(F^\lambda, L) \otimes F^\lambda$$

where $\mathrm{Spec}(\rho) \subseteq \widehat{G}$ denotes the set of irreducible representation classes that appears in L, and $g \in G$ acts in L via $I \otimes \pi^\lambda(g)$ on the summand for λ.

Now, assume $\mathcal{R} \subset \mathrm{End}(L)$ is a subalgebra such that

(i) $\mathcal{R}$ acts irreducibly on L,
(ii) if $g \in G$ and $T \in \mathcal{R}$, then $\rho(g)T\rho(g)^{-1} \in \mathcal{R}$ (so G acts on $\mathcal{R}$), and
(iii) the representation of G on $\mathcal{R}$ in (ii) is locally regular.

Remark. If dim $L < \infty$, then $\mathrm{End}(L)$ is the only option for $\mathcal{R}$. If L is infinite-dimensional, then there may exist many such algebras.

Let $\mathcal{R}^G = \{T \in \mathcal{R} : \rho(g)T = T\rho(g) \text{ for all } g \in G\}$. Since the action of $\mathcal{R}^G$ commutes with that of $\mathcal{A}[G]$, we have a representation of $(\mathcal{R}^G \times \mathcal{A}[G])$ on L. The duality theorem describes the decomposition of this representation.

Let $E^\lambda = \mathrm{Hom}_G(F^\lambda, L)$ for $\lambda \in \mathrm{Spec}(\rho)$. Then, E^λ is a module for $\mathcal{R}^G$ in a natural way as we can define an action of $\mathcal{R}^G$ on the component E^λ by $T \cdot u = T \circ u$ for $T \in \mathcal{R}^G$, $u \in \mathrm{Hom}_G(F^\lambda, L)$. Indeed, for $u \in E^\lambda, x \in L, g \in G$, we have

$$(Tu)(\pi^\lambda(g)x) = T(u(\pi^\lambda(g)x)) = T(\rho(g)u(x)) = \rho(g)(Tu(x)),$$

so $Tu \in E^\lambda$. Note that this action is compatible with the action of $\mathcal{R}^G \subset \mathcal{R}$ on L, since for any $T \in \mathcal{R}^G$, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_G(F^\lambda, L) \otimes F^\lambda & \xrightarrow{S_\lambda} & L \\ {\scriptstyle T\otimes I}\downarrow & & \downarrow{\scriptstyle T} \\ \mathrm{Hom}_G(F^\lambda, L) \otimes F^\lambda & \xrightarrow{S_\lambda} & L \end{array}$$

Hence, as a module for $\mathcal{R}^G \otimes \mathcal{A}[G]$, L decomposes as

$$L \cong \bigoplus_{\lambda\in\mathrm{Spec}(\rho)} E^\lambda \otimes F^\lambda.$$

Theorem 3.17 (Duality)**.** *With the above setup, for each $\lambda \in Spec(\rho)$, E^λ is an irreducible $\mathcal{R}^G$-module. If $E^\lambda \cong E^\mu$ as $\mathcal{R}^G$-modules, then $\lambda = \mu$.*

The duality theorem plays a central role in the representation and invariant theory of the classical groups. Here is an immediate consequence.

Corollary 3.18 (Duality Correspondence)**.** *Let σ be the representation of $\mathcal{R}^G$ on L and let $Spec(\sigma)$ denote the set of equivalence classes of the irreducible representations $\{E^\lambda\}$ on the algebra $\mathcal{R}^G$ that occur in L. Then, the following hold:*

(i) The representation (σ, L) is a direct sum of irreducible $\mathcal{R}^G$ modules, and each irreducible submodule E^λ occurs with finite multiplicity $\dim F^\lambda$.

(ii) The map $F^\lambda \to E^\lambda$ sets up a bijection between $Spec(\rho)$ and $Spec(\sigma)$.

The proof of the duality theorem requires the following lemma.

Lemma 3.19. *Let $X \subset L$ be a finite-dimensional G-invariant subspace of L. Then,* $\mathrm{Hom}_G(X, L) = \mathcal{R}^G\mid_X$.

Proof. Clearly, $\mathcal{R}^G\mid_X \subset \mathrm{Hom}_G(X, L)$, so we only need to show the other direction. Let $T \in \mathrm{Hom}_G(X, L)$. Let $\{v_1, \ldots, v_n\}$ be a basis for X. Since $\mathcal{R}$ acts irreducibly on L (this is assumption (i) of $\mathcal{R}$), by Jacobson's Density Theorem (Theorem 3.13), there exists an $r \in \mathcal{R}$ such that $Tv_i = rv_i$ for all $1 \le i \le n$. Hence $T = r\mid_X$. By the hypothesis, the action of G on $\mathcal{R}$ by conjugation is locally regular (this is assumption (ii),(iii) of $\mathcal{R}$). By Proposition 3.12, there exists a G-intertwining projection $p : \mathcal{R} \to \mathcal{R}^G$. Since the restriction $\mathcal{R} \to \mathrm{Hom}(X, L)$ also commutes with G actions, as X is G-invariant, we have $T = p(r)\mid_X$. $\square$

Proof of Theorem 3.17. First, we show that $E^\lambda = \mathrm{Hom}_G(F^\lambda, L)$ is irreducible under the action of $\mathcal{R}^G$. Let $T, S \in \mathrm{Hom}_G(F^\lambda, L)$, be non-zero elements, and we look for $r \in \mathcal{R}^G$ such that $r \circ T = S$. Note that by Schur's Lemma, TF^λ and SF^λ are isomorphic irreducible representations of G with class λ. By Lemma 3.19 (by setting $X = TF_\lambda$), there exists $r \in \mathcal{R}^G$ such that $r\mid_X$ gives such an isomorphism. Thus, $r \circ T : F^\lambda \to SF^\lambda$ is a G-module isomorphism. By Schur's Lemma ([11, Lemma 4.1.4]), there exists $c \in \mathbb{C}^\times$ such that $r\mid_X \circ T = cS$. Hence the action of $c^{-1}r$ sends T to S.

Now, suppose $\lambda \neq \mu$ for $\lambda, \mu \in \mathrm{Spec}(\rho)$. Suppose $\varphi : \mathrm{Hom}_G(F^\lambda, L) \to \mathrm{Hom}_G(F^\mu, L)$ is an $\mathcal{R}^G$-module map. We will show $\varphi = 0$. Let $T \in \mathrm{Hom}_G(F^\lambda, L)$ and set $S = \varphi(T)$. Let $U = TF^\lambda + SF^\mu$. Since $\lambda \neq \mu$, this sum is direct. Let $p : U \to SF^\mu$ be the projection relative to such direct sum decomposition. By Lemma 3.19, there exists $r \in \mathcal{R}^G$ such that $r\mid_U = p$. As $p \circ T : F^\lambda \xrightarrow{T} U \xrightarrow{p} SF^\mu = 0$, we have $r \circ T = 0$. Then $rS = r\varphi(T) = \varphi(rT) = 0$. But $rS = pS = S$, so $S = 0$. $\square$

Clearly, the Schur-Weyl Duality follows as a result of Theorem 3.17. By setting $G = \mathrm{GL}_n(\mathbb{C})$, $L = (\mathbb{C}^n)^{\otimes d}$, $\mathcal{R} = \mathrm{End}(L)$ given by the natural action of $G \times S_d$ on $(\mathbb{C}^n)^{\otimes d}$, we obtain

$$(\mathbb{C}^n)^{\otimes d} \cong \bigoplus_{\lambda \in S \subset \widehat{G}} E_d^\lambda \otimes F_n^\lambda,$$

where F_n^λ denotes an irreducible regular representation of $\mathrm{GL}_n(\mathbb{C})$ with highest weight λ, and E_d^λ denotes an irreducible representation of S_d determined by λ. The set S of highest weights λ that appear in the above decomposition are the partitions of d of length $\le n$, as described in §2.1.

4 Howe Duality

The theory presented in the previous sections implicitly assumes that all our groups exist over the complex numbers $\mathbb{C}$ (as $\mathcal{A}$ was defined over $\mathbb{C}$). However, as every number-theorist knows, $\mathbb{C}$ is boring[5]. So, we work with groups defined over local fields instead (the representation V is still over $\mathbb{C}$, though). There still exists a concept of "reductive" groups, and instead of regular representations, we work with "smooth" and "admissible" representations instead. The first section hopes to provide a brief background on this new class of representations.

[5] Or "beautiful, but well understood", as some others might suggest.

4.1 Smooth representations

In this section, we work with a non-Archimedean local field F, with unique maximal ideal $\mathfrak{p}$.

Definition 4.1. A *profinite group* is the inverse limit of a system of finite groups.

Definition 4.2. A *locally profinite group* is a topological group G such that every open neighbourhood of the identity in G contains a compact open subgroup of G.

Example 4.3. (i) The additive group F: $\mathfrak{p}^n$ for each $n \geq 1$ are compact open subgroups in F.

(ii) The multiplicative group $F^\times$: the congruence unit groups $U_F^n = 1 + \mathfrak{p}^n, n \geq 1$ are compact open subgroups.

(iii) The vector space $F^n = F \times \ldots \times F$ carries product topology, relative to which it is a locally profinite group. A special case is the matrix ring $M_n(F)$ is locally profinite under addition, in which multiplication is continuous.

(iv) The group $G = \mathrm{GL}_n(F)$ is the group of invertible matrices which is an open subset of $M_n(F)$; inversion is continuous so G is a topological group. The subgroups

$$K = \mathrm{GL}_n(\mathcal{O}_F), \qquad K_j = 1 + \mathfrak{p}^j M_n(\mathcal{O}_F), j \geq 1$$

are compact open and form a fundamental basis around 1 in $\mathrm{GL}_n(F)$.

With these groups defined, we now wish to consider its representations. In the previous section, we considered a specific subset of representations, namely locally regular representations, which yielded some significant and non-trivial results. We wish to find another analogous property for representations of these profinite groups.

Our profinite groups have a topology coming from the non-Archimedean topology on F, and we wish to leverage that. So, instead of considering abstract representations of the profinite group, we consider representations with certain continuity conditions. The most naïve way to do so would be to ask the homomorphism $\pi\colon G \to \mathrm{GL}(V)$ to be continuous. The problem here is that V is usually not finite-dimensional, so there is no clear choice of a topology on $\mathrm{GL}(V)$. So we rephrase this, by fixing a $v \in V$ and asking the function $G \to V : g \mapsto \pi(g)v$ to satisfy a continuity condition. Again, the correct topology on V is unclear since it is infinite-dimensional. So, we just give it the discrete topology. Thus, a representation (π, V) is *smooth* if for any $v \in V$ the function $G \to V : g \mapsto \pi(g)v$ is locally constant. Thus, we have the following definition:

Definition 4.4. Let G be a locally profinite group and let (π, V) be a representation of G. The representation (π, V) is *smooth* if for all $v \in V$, there exists a compact open subgroup K of G (depending on v) such that $\pi(x)v = v$ for all $x \in K$. Equivalently,

$$V = \bigcup_{K \subset G} V^K$$

where K ranges over compact open subgroups of G and V^K denotes $\pi(K)$-fixed vectors in V.

One indication that the above is a good definition is that when V is finite-dimensional, (π, V) being smooth is in fact equivalent to $\pi\colon G \to \mathrm{GL}(V)$ being continuous, where $\mathrm{GL}(V)$ carries the usual Archimedean topology.

Definition 4.5. A smooth representation (π, V) is called *admissible* if the space V^K is finite-dimensional, for each compact open subgroup K of G.

Admissibility ensures that most questions about the representation V can, in some sense, be reduced to questions about finite-dimensional spaces.

Similar to Proposition 3.10, there are some equivalent conditions for irreducibility.

Proposition 4.6. *Let G be a locally profinite group, and let (π, V) be a smooth representation of G. The following conditions are equivalent:*

(i) V is the sum of its irreducible G-subspaces;
(ii) V is the direct sum of a family of irreducible G-subspaces;
(iii) any G-subspace of V has a G-complement in V .

Proof. See [5, Chapter I §2.2 Proposition]. □

We say that V is *G-semisimple* if it satisfies the criteria of the above proposition. If G is a locally profinite group, then G can have many representations that are not semisimple. However, if G is locally profinite, K is compact open subgroup and if V is smooth, then V is K-semisimple (See Chapter I §2.2 Lemma [5]).

We state a lemma on irreducible admissible representations that shall be used in §4.4.

Lemma 4.7. *([22, Chapitre 2 III.4 Lemme]) Let G_1, G_2 be two locally profinite groups. Let (π_1, V_1) be an irreducible admissible representation of G_1, (π, V) a smooth representation of $G_1 \times G_2$. Suppose that $\bigcap_f ker(g) = \{0\}$ where f runs through $Hom_{G_1}(V_1, V_2)$. Then there exists a smooth representation (π_2, V_2) of G_2 unique up to isomorphism such that $\pi \cong \pi_1 \otimes \pi_2$.*

Proof. (Sketch) First, one shows that given two locally profinite groups G_1', G_2' with (π_1', V_2') an irreducible admissible representation of G_1', (π_2', V_2') a smooth representation of G_2', V' a $G_1' \times G_2'$-invariant subspace of $V_1' \otimes V_2'$, there exists a G_2'-invariant subspace $V_2'' \subset V_2'$ such that $V' = V_1' \otimes V_2''$. After proving this, we proceed as follows. For a G_1-submodule U, one considers the largest quotient on which G_1 acts trivially, denoted by $U[G_1]$. Then, one can set $V_2'' = (\check{V}_1 \otimes V)[G_1]$, where $\check{V}$ is the contragradient representation, and show that V injects into $V_1 \otimes V_2''$ as a $G_1 \times G_2$-module using the theory of Hecke algebras. Then, using the first fact, one obtains the required representation (π_2, V_2). For more details, see [22]. □

Now, we introduce some invariant measures on the space of locally profinite groups. Let G be a locally profinite group. Define the *support* of a function $f : G \to \mathbb{C}$ to be $\{g \in G : g(h) \neq 0\}$. Let $C_c^\infty(G)$ be the set of functions $f : G \to \mathbb{C}$ that are locally constant and of compact support (i.e., the support of f is a compact set). The group G acts on $C_c^\infty(G)$ by *left translation* λ and by *right translation* ρ:

$$\lambda_g f : x \longmapsto g(g^{-1}x), \quad \rho_g f : x \longmapsto f(xg), \quad x, g \in G, f \in C_c^\infty(G).$$

Both the G-representations $(C_c^\infty(G), \lambda)$ and $(C_c^\infty(G), \rho)$ are smooth.

Definition 4.8. A *right Haar integral* on G is a non-zero linear functional $I : C_c^\infty(G) \to \mathbb{C}$ such that
(i) $I(\rho_g f) = I(f), g \in G, f \in C_c^\infty(G)$, and
(ii) $I(f) \geq 0$ for any $f \in C_c^\infty(G), f \geq 0$.

One can define a *left Haar integral* similarly, using left translation λ instead of a right translation. Further, one can show that G possesses a right Haar integral that is essentially unique, see [5, Chapter 1 §3.2 Proposition].

Definition 4.9. Let I be a left Haar integral on G and let $S \neq \varnothing$ be a compact open set. Set Γ_S to be the characteristic function of S. We define $\mu_G(S) = I(\Gamma_S)$. Then $\mu(G)(S) > 0$ and satisfies $\mu_G(gS) = \mu_G(S), g \in G$. We refer to μ_G as the *left Haar measure* on G.

Notation. The relation with the integral is expressed via the notation

$$I(f) = \int_G f(g) d\mu_G(g), \quad f \in C_c^\infty(G).$$

Often, the following shorthand will also be used:

$$\int_G \Gamma_S(x) f(x) d\mu_G(x)(x) = \int_S f(x) d\mu_G(x).$$

For more detail on the Haar measure and its properties, see [5, Chapter 1 §3].

4.2 The Heisenberg group

We begin this section with the following hypothesis: Let F be our non-Archimedean local field and assume that all our representations are over $\mathbb{C}$ and smooth.

We will denote by $S(W)$ the space of all locally constant compactly supported functions on any vector space W (also called the Schwartz space, denoted as $C_c^\infty(W)$ in the previous section).

Definition 4.10. Let W be a finite-dimensional vector space over dimension $2n$ over F with non-degenerate alternating form $\langle\ ,\ \rangle$. Such a vector space is called a *symplectic space*.

Definition 4.11. A *central extension* of a group G by A is a short exact sequence of groups

$$1 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1$$

such that $A \subseteq Z(E)$, where $Z(E)$ is the centre of E.

Definition 4.12. A *totally isotropic subspace* of a vector space is a vector subspace on which the bilinear form vanishes.

Definition 4.13. Let W be a symplectic space. The *Heisenberg group* $H(W)$ is a non-trivial central extension of W by F and is defined to be the group of pairs $H(W) := W \oplus F = \{(w,t) : w \in W, t \in F\}$ with the law of multiplication

$$(w_1,t_1)\cdot(w_2,t_2) = \left(w_1 + w_2, t_1 + t_2 + \frac{1}{2}\langle w_1, w_2\rangle\right). \tag{2}$$

$H(W)$ is a locally profinite group. To see this, note that $H(W) = W \oplus F \cong F^{2n} \oplus F = \prod_{i=1}^{2n+1} F^i$ as vector spaces. As the product of open sets is open and the product of compact sets is compact in the product topology, we have that every open neighbourhood of $0 \times \ldots \times 0$ contains an open compact set (as each copy of F is locally profinite as in Example 4.3). To see that this compact set is indeed a subgroup, note that the multiplication defined above is still closed in this set, as for $x \in \mathfrak{p}^k$ and $y \in \mathfrak{p}^m$, $k \geq m$ we have $x + y \in \mathfrak{p}^m$ and $\mathfrak{p}^k \subseteq \mathfrak{p}^m$. So, if needed, we may take our compact set to be $\mathfrak{p}^m \times \ldots \times \mathfrak{p}^m$ for the smallest such m where this addition stays closed. We state some other straightforward facts about the Heisenberg group:

1. There is an exact sequence $0 \to F \to H(W) \to W \to 0$.
2. The commutator subgroup is equal to the centre which is F, i.e., $[H(W), H(W)] = Z(H(W)) = F$.
3. All one-dimensional representations of W factor through $H(W)$.

The goal is to construct infinite-dimensional smooth representations of $H(W)$. The first representation we construct is called the Schrödinger representation, which has many applications in physics.

Fix a decomposition $W = W_1 \oplus W_2$ where W_1, W_2 are maximal totally isotropic subspaces of W. Such a decomposition will be called a *complete polarisation* of W. Let $\zeta : F \to H(W)$ denote the injection $\zeta(t) \mapsto (0,t)$. Let $\delta : W \to H(W)$ denote the injection $\delta(w) \mapsto (w, 0)$. Let ψ be an additive character of F. Define the representation ρ_ψ of $H(W)$ on $S(W_1)$ as follows:

$$\begin{aligned} \rho_\psi(w_1)f(x) &= f(x + w_1), \quad f \in S(W_1), x_1, w_1 \in W_1 \\ \rho_\psi(w_2)f(x) &= \psi\left(\langle x, w_2\rangle\right) f(x), \quad f \in S(W_1), x \in W_1, w_2 \in W_2 \\ \rho_\psi(t)f(x) &= \psi(t)f(x), \quad t \in F, x \in W_1 \end{aligned} \tag{3}$$

Note that this does describe an action on $H(W)$, as any element in $w \in W = W_1 \oplus W_2$ can be written as $w_1 + w_2$. Then, for any $t \in F$, we have $(w,t) = (w_1 + w_2, t)$, which implies $(w_1, 0)\cdot(w_2, 0)\cdot(0,t) = (w_1 + w_2, t + 1/2\langle w_1, w_2\rangle)$. So, $\rho_\psi((w,s)) = \rho_\psi(w_1)\rho_\psi(w_2)\rho_\psi(t)$ where $t = s - 1/2\langle w_1, w_2\rangle$ and each ρ_ψ is given by (3).

We can construct some more general representations of $H(W)$. Let $A \subset W$ be a closed subgroup. Let $A^\perp = \{y \in W : \psi(\langle x, y\rangle) = 1 \text{ for all } x \in A\}$ and assume $A = A^\perp$. Examples of subspaces with this property include:

(i) $A = W_1$ a maximal totally isotropic subspace of W.
(ii) $A = L = \sum_{i=1}^n \mathcal{O}_F e_i \oplus \sum_{i=1}^n \mathcal{O}_F f_i$ where $\{e_i, f_i\}$ is a symplectic basis of W i.e., $\langle e_i, e_j\rangle = 0$, $\langle f_i, f_j\rangle = 0$, $\langle e_i, f_j\rangle = \delta_{ij}$, and $\mathcal{O}_F$ is the ring of integers for F.

Define the subgroup $A_F = A \times F \subset H(W)$.We can extend ψ of F to A_F by $\psi((a,t)) = \psi(t)$ for all $a \in A, t \in F$. Indeed, as $A = A^\perp$, we have

$$\begin{aligned} &(a_1, t_1) \cdot (a_2, t_2) = (a_1 + a_2, t_1 + t_2 + 1/2\langle a_1, a_2\rangle) = (a_1 + a_2, t_1 + t_2) \\ \Rightarrow \quad &\psi((a_1, t_1))\psi((a_2, t_2)) = \psi(t_1)\psi(t_2) = \psi((a_1 + a_2, t_1 + t_2)) \end{aligned}$$

so ψ is a homomorphism on A_F. Fix such an extension and call it ψ_A.

Then, let S_A denote the space of function on $H(W)$ such that

(i) $f(ah) = \psi_A(a)f(h)$ for all $a \in A_F, h \in H(W)$,
(ii) $f(hl) = f(h)$ for all $l \in L$, a lattice in W, and $h \in H(W)$.

Then, S_A is compactly supported modulo A_F. $H(W)$ acts on S_A via right translations and this representation is smooth because of (ii).

We have now constructed many examples of smooth representations of $H(W)$ on which the centre acts via ψ. Perhaps to the reader's relief – or dismay – it turns out that all these representations are isomorphic. A foundational result by Stone and von Neumann goes as follows:

Theorem 4.14. *Up to isomorphism, there exists a unique smooth irreducible representation $\rho_\psi, S)$ of $H(W)$ with central character ψ, i.e., such that $\rho_\psi((0,t)) = \psi(t) \cdot id_S$.*

Proof. [22, Chapitre 2, Theoreme I.2 pp 28-31]. □

The theorem implies that each character determines a unique isomorphism class of representations, so there is limited variety at the level of isomorphism classes, as characters are well understood. However, within a given isomorphism class, there are still things of interest. Although the representations are isomorphic, they may admit multiple, distinct constructions. Thus, the focus shifts to understanding the various ways in which a given representation can be realised. We will refer to these as *models* of (ρ_ψ, S).

4.3 The Metaplectic group and the Weil representation

We begin with the symplectic vector space $(W, \langle\ ,\ \rangle)$ of dimension $2n$ over a non-Archimedean local field of characteristic not equal to 2. Consider the symplectic group over W, i.e.,

$$\mathrm{Sp}(W) = \{g \in \mathrm{GL}(W) : \langle gx, gy\rangle = \langle x, y\rangle\}.$$

Then, $\mathrm{Sp}(W)$ acts on $\mathrm{Aut}(H(W))$ by $g \cdot (w,t) = (g \cdot w, t)$. The action is trivial on the centre of $H(W) = F$. The representation (ρ_ψ^g, S) given by $\rho_\psi^g(h) = \rho_\psi(gh)$ again has central character ψ and so is isomorphic to (ρ_ψ, S) by Stone-von Neumann. Now, let $g \in \mathrm{Sp}(W)$. Fix a Haar measure on the vector space $W/\ker(1-g)$. One can check that the function on W give by $w \longmapsto \psi(\langle w, g \cdot w\rangle/2)$ is constant on the classes modulo $\ker(1-g)$. Recall that $\delta : W \to H(W)$ denotes the injection $\delta(w) = (w, 0)$.

If F is a finite, one can define an endomorphism M, or $M(g)$, on S by

$$Ms = \int_{W/\ker(1-g)} \psi(\langle w, g \cdot w\rangle/2)\rho_\psi(\delta \circ (1-g) \cdot w)s\ dw$$

for all $s \in S$. If F is a local field, let L be a ring in $W/\ker(1-g)$. For $s \in S$, we define an element $M_L s \in S$ by

$$M_L s = \int_L \psi(\langle w, g \cdot w\rangle/2)\rho_\psi(\delta \circ (1-g) \cdot w)s\ dw.$$

This map $M(g)$ satisfies some properties:

Lemma 4.15. *For all $s \in S$, there exists a lattice $L_s \subset W/ker(1-g)$ and an element $Ms \in S$ such that if i) L is a lattice in in $W/ker(1-g)$, ii) $L_s \subset L$, then $M_L s = Ms$.*

Proof. See [22, Chapitre 2 II.2 Lemme]. ☐

Lemma 4.16. *For all $g \in \mathrm{Sp}(W)$, $h \in H(W)$, we have*

$$M(g)\rho_\psi(h) = \rho_\psi(gh)M(g). \tag{4}$$

Proof. Let $M = M(g)$, and suppose $h = \delta(w_0)$, and $s \in S$. For a sufficiently large lattice L, we have

$$\begin{aligned}\rho_\psi(\delta(gw_0))Ms &= \rho_\psi(\delta(gw_0))M_L s \\ &= \int_L \psi(\langle w, g\cdot w\rangle/2)\rho_\psi(\delta(gw_0)\delta\circ(1-g)w)s\ dw.\end{aligned}$$

As

$$\begin{aligned}\delta(gw_0)\delta\circ(1-g)w &= \delta(gw_0+(1-g)w)\zeta(\langle gw_0,(1-g)w\rangle/2) \\ &= \delta((1-g)(w-w_0)+w_0)\zeta(\langle gw_0,(1-g)w\rangle/2) \\ &= \delta(1-g)(w-w_0)\delta(w_0)\zeta(\langle gw_0,(1-g)w\rangle/2+\langle w_0,(1-g)(w-w_0)\rangle),\end{aligned}$$

and

$$\langle w, gw\rangle + \langle gw_0,(1-g)w\rangle + \langle w_0,(1-g)(w-w_0)\rangle = \langle w-w_0, g(w-w_0)\rangle,$$

we have

$$\rho_\psi(\delta(gw_0))Ms = \int_L \psi(\langle w-w_0, g(w-w_0)/w\rangle \rho_\psi(\delta\circ(1-g)(w-w_0))\rho_\psi(\delta w_0)s\ dw.$$

For L sufficiently large, we may assume $w_0 \in L$. On carrying out a change of variable $w-w_0 \rightsquigarrow w$, the right hand side becomes $M_L \circ \rho_\psi(\delta(w_0))s$. For L sufficiently large, by Lemma 4.15, this is $M(g)\circ\rho_\psi(h)$. ☐

Essentially, we have constructed an automorphism $M(g): S \to S$ such that

$$M(g)\rho_\psi(h)M(g)^{-1} = \rho_\psi(gh) = \rho_\psi^g(h).$$

$M(g)$ is only unique up to a scalar in $\mathbb{C}^\times$, so $M(g_1)M(g_2) \neq M(g_1g_2)$. However, we can define a projective representation

$$\mathrm{Sp}(W) \to \mathrm{PGL}(S) = \mathrm{GL}(S)/\mathbb{C}^\times, \quad g \mapsto M(g).$$

As one tends to with projective representations, we attempt to "de-projectivise" it.

Definition 4.17. The *metaplectic group* is defined to be

$$\widetilde{\mathrm{Sp}(W)}_\psi = \mathrm{Mp}(W) := \{(g, M(g)) \in \mathrm{Sp}(W)\times\mathrm{GL}(S) : (4) \text{ holds }\}.$$

Definition 4.18. The metaplectic group sits in the central extension

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \mathrm{Mp}(W)_\psi \longrightarrow \mathrm{Sp}(W) \longrightarrow 1,$$

such that M may be lifted to a representation ω_ψ of $\mathrm{Mp}(W)$, that is $\omega_\psi((g, M(g))) = M(g)$. This genuine representation is called the *Weil representation*, the *metaplectic representation* or the *oscillator representation*.[6]

[6] If you have ever wondered how multiple names for the same object arise, in the words of Howe, "Until now, this representation has enjoyed a rather ad hoc name 'Weil Representation'. However, in view of the increasing evidence that it is fundamental object [...], I am so bold as to attempt to rechristen it: in this paper we shall refer to this representation as the oscillator representation." - [16]

Given that there are many explicit realisations of the representation (ρ_ψ, S), there are many explicit realisations of the Weil representation as well. We will give an explicit description of the lattice and Schrödinger model of the Weil representation.

For the Schrödinger model, we let $W = W_1 \oplus W_2$ be a complete polarisation. Let elements in $\mathrm{Sp}(W)$ be written as matrices with respect to a basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ where $e_i \in W_1$ and $f_i \in W_2$ and $\langle e_i, f_i \rangle = \delta_{ij}$. The generators of $\mathrm{Sp}(W)$ are given by

(i) $\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}$, $A \in \mathrm{GL}(W_1)$.

(ii) $\begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix}$, $B = B^t$,

(iii) $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$

Then, the action of these generators in the Weil representation on $S(W_1)$ is given by

$$\omega_\psi \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} f(x) = \mid \det A \mid^{1/2} f({}^tAx)$$
$$\omega_\psi \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} f(x) = \psi\left(\frac{{}^txBx}{2}\right) f(x)$$
$$\omega_\psi \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} f(x) = \gamma \hat{f}(s)$$

where

$$\hat{f}(x) = \int_{F^n} f(y)\psi\left(\sum_{i=1}^n x_i y_i\right) dy,$$

γ is an eighth root of unity, $\hat{f}(x)$ is the Fourier transform and the Haar measure dy is chosen so that $\hat{\hat{f}}(x) = f(-x)$. For more details, see [24, §2].

For the lattice model, let A be a finitely generated $\mathcal{O}_F$-module of maximal rank in W such that $A = A^\perp$. Then, in the lattice model, we have $(g, M(g)) \in \mathrm{Mp}(W)$ acting on $f \in S(W_1)$ as

$$\left(M(g)\cdot f\right)(w) = \sum_{a \in A/gA\cap A} \psi\left(\frac{\langle a, w\rangle}{2}\right) f\left(g^{-1}(a+w)\right).$$

4.4 Reductive dual pairs and the Local Theta Correspondence

In this section, we define a reductive dual pair. We follow the exposition given in Howe's unpublished notes on the algebraic preliminaries of the oscillator representation [15].

Definition 4.19. Let Γ be a group and let (G, G') be a pair of subgroups in Γ. We will say (G, G') form a *dual pair* of subgroups of Γ if G is the centraliser in Γ of G' and vice-versa.

Note that one can find an abundance of dual pairs. For example, let $G \subset \Gamma$ be any subgroup. Then, set $G' = \mathrm{Cent}_\Gamma(G)$ and $G'' = \mathrm{Cent}_\Gamma(G')$. Then, (G', G'') is a dual pair in Γ.

We can refine the concept further when $\Gamma = \mathrm{Sp}(W)$.

Definition 4.20. A *reductive dual pair* (G, G') in $\mathrm{Sp}(W)$ is a pair of subgroups G, G' in $\mathrm{Sp}(W)$ such that G and G' are reductive groups and

$$\mathrm{Cent}_{\mathrm{Sp}(W)}(G) = G' \quad \text{and} \quad \mathrm{Cent}_{\mathrm{Sp}(W)}(G') = G.$$

Some examples of reductive dual pairs include:

(i) $(\mathrm{Sp}(W), \{\pm 1_W\})$, a trivial example (Type I).

(ii) Let $\mathbf{e}$ be a standard basis for W as in §4.2, and let T be the subgroup of $\mathrm{Sp}(W)$ which acts diagonally with respect to this basis. Then $\mathrm{Cent}_{\mathrm{Sp}(W)}(T) = T$, so that (T, T) is a reductive dual pair (Type II).

(iii) If (G_1, G_1') and $G_2, G_2')$ are dual pairs in $\mathrm{Sp}(W_1)$ and $\mathrm{Sp}(W_2)$, then $(G_1 \times G_2, G_1' \times G_2')$ is a reductive dual pair in $\mathrm{Sp}(W_1 \oplus W_2)$.

Definition 4.21. A reductive dual pair (G, G') in $\mathrm{Sp}(W)$ will be called *irreducible* if one cannot decompose W as the direct sum of two symplectic subspaces, each of which is invariant under both G and G'.

We know pretty much everything one can know about irreducible dual pairs. They can be explicitly constructed in only two ways, that we describe below, following the exposition in [16, pp. 26-28].

Type I:

Let $(V_0, \langle\ ,\ \rangle_0)$ be a symplectic vector space and $(U_0, (\ ,\ )_0)$ be an orthogonal vector space with respective inner product. Set $V = V_0 \otimes U_0$. On V, define a bilinear form $\langle\ ,\ \rangle$ by $\langle v \otimes u, v' \otimes u\rangle = \langle v, v'\rangle_0 \cdot (u, u')_0$. This makes V a symplectic vector space. Define the maps

$$j_1 : \mathrm{GL}(V_0) \to \mathrm{GL}(V), \quad j_1(g)(v \otimes u) = g(v) \otimes u.$$

Similarly, define

$$j_2 : \mathrm{GL}(U_0) \to \mathrm{GL}(V), \quad j_2(g)(v \otimes u) = v \otimes g(u).$$

Now, let $O(U_0, (\ )_0) := O(U)$ be the group of isometries of $(\ ,\ )_0$ on U_0. Then, $(j_1(\mathrm{Sp}(V_0)), j_2(O(U)))$ is a *Type I* reductive dual pair in $\mathrm{Sp}(V)$.

Type II:

Type II is slightly more complicated to define. Let Y_1 and Y_2 be vector spaces. Set

$$Y := Y_1 \otimes Y_2, \qquad \text{and } V := Y \oplus Y^* = (Y_1 \otimes Y_2) \oplus (Y_1 \otimes Y_2)^*.$$

Define the symplectic form $\langle\ ,\ \rangle$ on V by

$$\langle (y, y^*), (z, z^*)\rangle = z^*(y) - y^*(z).$$

Then, define

$$\alpha : \mathrm{GL}(Y) \to \mathrm{Sp}(V, \langle\ ,\ \rangle), \quad \alpha(g)(y, y^*) = \big(g(y), g^{*-1}(y^*)\big)$$

where g^* is the usual adjoint of g, given by the formula $g^*(y^*)(y) = y^*(g(y))$. Define injections

$$\begin{aligned} j_1 : \mathrm{GL}(Y_1) \to \mathrm{GL}(Y), \quad & j_1(g_1)(y_1 \otimes y_2) = g_1(y_1) \otimes y_2, \quad && \text{for all } y_i \in Y_i, g_1 \in \mathrm{GL}(Y_1),\\ j_2 : \mathrm{GL}(Y_2) \to \mathrm{GL}(Y), \quad & j_2(g_2)(y_1 \otimes y_2) = y_1 \otimes g_2(y_2), \quad && \text{for all } y_i \in Y_i, g_2 \in \mathrm{GL}(Y_2). \end{aligned}$$

Then, $(\alpha(j_1(\mathrm{GL}(Y_1))), \alpha(j_2(\mathrm{GL}(Y_2))))$ is a reductive dual pair of *Type II* in $\mathrm{Sp}(V)$.

For a proof that these are the only two ways irreducible dual reductive pairs arise, see [15, Proposition 6.1].

We now provide two lemmas needed to state the Howe Duality and the Local Theta Correspondence.

Lemma 4.22. *If $g_1, g_2 \in \mathrm{Sp}(W)$ commute, then $M(g_1)M(g_2) = M(g_2)M(g_1)$. In other words, if two elements of $\mathrm{Sp}(W)$ commute, then any two inverse images in the metaplectic group of these two elements commute too.*

Proof. Let $s \in S$ and L be a lattice in $W/\ker(1 - g_2)$. If L is sufficiently large, by Lemma 4.15, we have

$$\begin{aligned} M(g_1)M(g_2)s &= M(g_1)M_L(g_2)s \\ &= \int_L \psi(\langle w, g_2 w\rangle/2) M(g_1)\rho_\psi(\delta \circ (1 - g_2)w)s\ dw \\ &= \int_L \psi(\langle w, gw\rangle/2)\rho_\psi(\delta \circ g_1(1 - g_2)w)M(g_1)s\ dw, \end{aligned}$$

where the last equality follows from Lemma 4.16. We perform a change of variables and set $w = g^{-1}w'$. The Jacobian then becomes

$$| \det \left(g^{-1} \mid_{W/\ker(1-g_2)}\right) | = | \det \left(g_1 \mid_W\right) |^{-1} | \det(g_1 \mid_{\ker(1-g_2)}) | \ .$$

The first term is 1 as $g_1 \in \mathrm{Sp}(W)$. Using the explicit commutant description of g_2 ([3, §IV.2]), one obtains that the second term must also be 1. Since g_1, g_2 commute, we obtain

$$\begin{aligned} M(g_1)M(g_2)s &= \int_L \psi(\langle w, g_2 w\rangle)\rho_\psi(\delta \circ (1-g_2)w)M(g_1)s \ dw \\ &= M_{g_1 L}(g_2)M(g_1)s. \end{aligned}$$

If L is large enough, from Lemma 4.15, we obtain $M(g_2)M(g_1)$. □

Let (G, G') be a reductive dual pair in $\mathrm{Sp}(W)$. Let $\widetilde{G}, \widetilde{G'}$ be the inverse image of G, G' in the metaplectic group $\mathrm{Mp}(W)$, so that these groups sit in the central extensions

$$0 \to \mathbb{C}^\times \to \widetilde{G} \to G \to 0 \quad \text{and} \quad 0 \to \mathbb{C}^\times \to \widetilde{G'} \to G' \to 0.$$

So, there is a natural homomorphism $j : \widetilde{G} \times \widetilde{G'} \to \widetilde{G \cdot G'} \subset \mathrm{Mp}(W)$.

For an additive character ψ, we may consider the pullback $(j^*(\omega_\psi), S)$ of the Weil representation (ω_ψ, S) to $\widetilde{G} \times \widetilde{G'}$, so we have

$$\omega_\psi \circ j : \widetilde{G} \times \widetilde{G'} \xrightarrow{j} \widetilde{G \cdot G'} \hookrightarrow \mathrm{Mp}(W) \xrightarrow{\omega_\psi} \mathrm{GL}(S),$$

which is essentially the restriction of ω_ψ to $\widetilde{G} \times \widetilde{G'}$. Furthermore, as $\mathbb{C}^\times \hookrightarrow \mathrm{Mp}(W)$ by $t \mapsto (1, t)$, we have $\omega_\psi((1,t)) = t \cdot id_S$, and so the central character on $\widetilde{G}$ and $\widetilde{G'}$ both act on S by the identity character.

The heuristic underpinning of the Howe duality is that the Weil representation, being relatively "small," should admit a well-behaved decomposition into irreducibles when restricted to the subgroup $\widetilde{G} \times \widetilde{G'}$, formed by mutual commutants. As there are many non-trivial extensions in the category of smooth or admissible representations – often depending on how the character is extended – it is more natural to study suitable quotients of $(j^*(\omega_\psi), S)$, rather than submodules or direct summands. Evidently, only those irreducible representations π of $\widetilde{G}$ (or $\widetilde{G'}$) satisfying $\pi(t) = t \cdot \mathrm{id}_\pi$ can arise in this context.

Now, let (π, V_π) be an irreducible admissible representation of $\widetilde{G}$ for which $\pi(z) = z \cdot id_\pi$. Let

$$\begin{aligned} S(V_\pi) = S(\pi) :=& \text{ maximal quotient of } S \text{ on which } \widetilde{G} \text{ acts as a multiple of } \pi \\ =& \text{ largest quotient of } S \text{ which is } \pi\text{-isotypic} \ . \end{aligned}$$

Let $N(\pi) = \bigcap_{\lambda \in \mathrm{Hom}_{\widetilde{G}}(S,\pi)} \ker\lambda$ be the kernel of all $\widetilde{G}$-equivariant maps from S to V_π. Then, $S(\pi) = S/N(\pi)$. We claim that $S(\pi)$ is also a $\widetilde{G'}$-module. Indeed, as given $g \in \widetilde{G}$ and $g' \in \widetilde{G'}$, define $\lambda^{g'} : S \to V_\pi, \ \lambda^{g'}(s) = \lambda(g \cdot s)$. Then $\lambda^{g'}(g \cdot s) = \lambda(g' \cdot g \cdot s) = \lambda(g' g \cdot s) = g\lambda(g' \cdot s)$, i.e., the action of $\widetilde{G}$ commutes with $\widetilde{G'}$. This makes $S(\pi)$ into a $\widetilde{G} \times \widetilde{G'}$-module.

From Lemma 4.7, we must have $S(\pi) = \pi \otimes \Theta_\psi(\pi)$ where $\Theta_\psi(\pi)$ is unique up to isomorphism. We call $\Theta_\psi(\pi)$ the *big theta lift* of π. If π does not occur as a quotient of (ω_ψ, S), then set $S(\pi)$ and $\Theta_\psi(\pi) = 0$. We can now state the Howe Duality Principle.

Theorem 4.23. *For any irreducible admissible representation π of $\widetilde{G}$,*

(i) Either $\Theta_\psi(\pi) = 0$ or $\Theta_\psi(\pi)$ is an admissible representation of $\widetilde{G'}$ of finite length.

(ii) If $\Theta_\psi(\pi) \neq 0$, there is a unique $\widetilde{G'}$-invariant submodule $\Theta^0_\psi(\pi)$ of $\Theta_\psi(\pi)$ such that the quotient

$$\theta_\psi(\pi) := \Theta_\psi(\pi)/\Theta^0_\psi(\pi)$$

is irreducible. The irreducible admissible representation $\theta_\psi(\pi)$ of $\widetilde{G'}$ is uniquely determined by π. If $\Theta_\psi(\pi) = 0$, set $\theta_\psi(\pi) = 0$.

(iii) If $\theta_\psi(\pi_1)$ and $\theta_\psi(\pi_2)$ are non-zero and isomorphic, then $\pi_1 \cong \pi_2$.

Definition 4.24. Let

$$\mathrm{Howe}_\psi(\widetilde{G};\widetilde{G'}) = \{\pi \in \mathrm{Irr}(\widetilde{G}) \mid \theta_\psi(\pi) \neq 0\}$$

be the set of all (isomorphism classes of) irreducible admissible representations of $\widetilde{G}$ such that $\mathrm{Hom}_{\widetilde{G}}(S,\pi) \neq 0$. Note, this set can depend on both the dual pair (G, G') and on the choice of ψ. The Howe Duality principle asserts that the map $\pi \longmapsto \theta_\psi(\pi)$ defines a bijection

$$\mathrm{Howe}_\psi(\widetilde{G};\widetilde{G'}) \longleftrightarrow \mathrm{Howe}_\psi(\widetilde{G'};\widetilde{G})$$

referred to as the *local theta correspondence*.

The proof of this statement is unfortunately beyond the scope of this report. The local theta correspondence has many applications including being an important tool in the theory of automorphic forms – one famous application being the construction of counterexamples to the generalized Ramanujan–Petersson conjecture [17], others being cases of the local Langlands conjectures [9] and Gan-Gross-Prasad conjectures [10]. However, we would like to let the reader know that despite the number theorist's usual impatience with $\mathbb{C}$, it turns out the archimedean case is not entirely devoid of interest: Howe initially formulated his conjecture over $\mathbb{R}$ and $\mathbb{Q}_p$. Over $\mathbb{R}$, one works with Harish-Chandra modules of the unitary metaplectic representation. The $(\mathfrak{g}, \widetilde{K})$ module of the oscillator representation of $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ is called the Fock module. To see a description of the Harish-Chandra module in the Fock module, see [24, §5.]. To see a description of the oscillator representation of the complex Heisenberg group and symplectic Lie algebras, see [1].

Over the p-adics, the Howe duality conjecture has been established completely, thanks to the works of Waldspurger [25], Minguez [20], Gan and Takeda [10], and Gan and Sun [8]. The case for $(O(p,q), \mathrm{Sp}(2n,\mathbb{R}))$ has been established by Moeglin in [21]. In[2], Adams talks about reductive dual pairs in the complex case.

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