Duality Theorems: From Schur-Weyl to Howe

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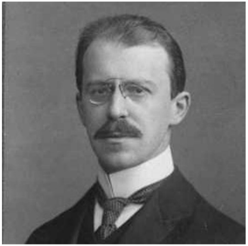

Issai Schur

Hermann Weyl

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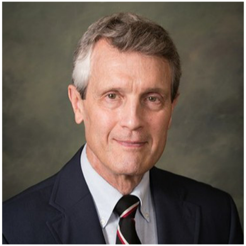

Roger Howe

Definition (Representation of a group)

Let G be a group and V a vector space over a field k. A *representation* of G on V can be defined equivalently in the following ways:

1. **Homomorphism definition.** A representation is a group homomorphism
$$\rho : G \to \mathrm{GL}(V),$$
where $\mathrm{GL}(V)$ is the group of invertible linear transformations of V.
2. **Group action definition.** A representation is a linear action of G on V, i.e. a map
$$G \times V \to V, \quad (g, v) \mapsto g \cdot v,$$
such that $e \cdot v = v$ and $(gh) \cdot v = g \cdot (h \cdot v)$ for all $g, h \in G$, $v \in V$.
3. $\mathbb{C}[G]$**-module definition.** For a finite dimensional representation on a vector space V, V becomes a $\mathbb{C}[G]$-module where the action of $g \in G$ on a vector $v \in V$ is defined by extending the group's action linearly
$$\left(\sum_{g \in G} a_g g \right) \cdot v = \sum_{g \in G} g_g (g \cdot v).$$

Irreducible Representations of S_n

Let S_n denote the symmetric group of n elements.
One can categorise all irreducible representations of S_n.

There is the following one-one correspondence between irreducible representations of S_n and partitions of n:

$$\left\{ \begin{array}{c} \text{irreducible} \\ \text{representations} \\ \text{of } S_n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{conjugacy} \\ \text{classes of } S_n \end{array} \right\} \xleftrightarrow{\text{cycle types}} \left\{ \begin{array}{c} \text{partitions of} \\ n \end{array} \right\}$$

Definition:

Let $G = \mathrm{GL}_n(\mathbb{C})$. Let $\rho : G \to \mathrm{GL}(V)$ be finite dimensional representation. Identify $\mathrm{GL}(V) = \mathrm{GL}_m(\mathbb{C})$. Let $g = (g_{ij}) \in G$.

Then $\rho(g) = (\rho(g)_{kl})$ is a matrix. The **regularity** of (ρ, V) means that the matrix coefficients $\rho(g)_{kl}$ are polynomials in g_{ij} and in $\det(g)^{-1}$. If $\det(g)^{-1}$ does not appear, it is called a **polynomial** representation.

Polynomial representation: Example

Let $G = \mathrm{GL}_2(\mathbb{C})$. Let V be a 2-dimensional $\mathbb{C}$ vector space, with basis e_1, e_2. The symmetric square $\mathrm{Sym}^2 V$ is a representation of $\mathrm{GL}_2(\mathbb{C})$. With respect to the basis $e_1 e_2, e_1^2, e_2^2$ of $\mathrm{Sym}^2 V$, the matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

acts on $\mathrm{Sym}^2 V$ as

$$\begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad + bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}.$$

Irreducible Representations of $\mathrm{GL}_n(\mathbb{C})$

One can categorise all irreducible representations of $\mathrm{GL}_n(\mathbb{C})$. Using some ✨ Lie algebra magic ✨, one can deduce

$$\left\{ \begin{array}{c} \text{irreducible} \\ \text{polynomial} \\ \text{representations} \\ \text{of } \mathrm{GL}_n(\mathbb{C}) \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{highest} \\ \text{weights } \lambda \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} (\lambda_1, \ldots, \lambda_n), \\ \text{with } \lambda_i \in \mathbb{Z}_{\geq 0} \\ \text{and} \\ \lambda_1 \leq \ldots \leq \lambda_n \end{array} \right\}$$

Schur-Weyl Duality: Set-up

Both $\mathrm{GL}_n(\mathbb{C})$ and S_d act naturally on $(\mathbb{C}^n)^{\otimes d}$.
$\mathrm{GL}_n(\mathbb{C})$ acts diagonally via

$$g \cdot (v_1 \otimes \ldots \otimes v_d) = g \cdot v_1 \otimes g \cdot v_2 \otimes \ldots \otimes g \cdot v_d.$$

S_d acts by permuting the entries as

$$(v_1 \otimes \ldots \otimes v_d) \cdot w = v_{w(1)} \otimes v_{w(2)} \otimes \ldots \otimes v_{w(d)}.$$

So, we may view $(\mathbb{C}^n)^{\otimes d}$ as a $(\mathrm{GL}_n(\mathbb{C}) \times S_d)$-module.

Theorem:

The $\mathrm{GL}_n(\mathbb{C}) \times S_d$-module $(\mathbb{C}^n)^{\otimes d}$ decomposes as follows:

$$(\mathbb{C}^n)^{\otimes d} = \bigoplus_{\lambda} \pi_\lambda^{\mathrm{GL}_n(\mathbb{C})} \otimes \pi_\lambda^{S_d},$$

where λ runs through the partitions of d of length $\leq n$; $\pi_\lambda^{\mathrm{GL}_n(\mathbb{C})}$ is the irreducible representation of $\mathrm{GL}_n(\mathbb{C})$ of highest weight λ. $\pi_\lambda^{S_d}$ is the irreducible representation of S_d corresponding to the partition λ.

Question: Why is this special?

$\mathrm{GL}_n(\mathbb{C}) \times S_d$ acts on $(\mathbb{C}^n)^{\otimes d}$. This would mean that the action would break down into indecomposables. However, it is not obvious that:

- The $\mathbb{C}[S_d]$ and $\mathrm{GL}_n(\mathbb{C})$-modules can *both* sum over partitions of d of length $\leq n$.
- The decomposition of both the $\mathbb{C}[S_d]$ and $\mathrm{GL}_n(\mathbb{C})$-modules are *irreducible.*

Schur-Weyl Duality: Examples

- $d = 2$, $n \geq 1$:

$$(\mathbb{C}^n)^{\otimes 2} = (\mathrm{Sym}^2\mathbb{C}^n \otimes \mathrm{triv}) \oplus \left(\bigwedge^2 \mathbb{C}^n \otimes \mathrm{sgn}\right).$$

- $d = 3$, $n \geq 1$:
Denote

$$V_{(2,1)} = \langle v_1 \otimes v_2 \otimes v_3 - v_3 \otimes v_2 \otimes v_1 + v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_1 \otimes v_2, v_2 \otimes v_1 \otimes v_1 - v_1 \otimes v_2 \otimes v_2 \rangle.$$

Then,
$(\mathbb{C}^n)^{\otimes 3} \cong (\mathrm{Sym}^3\mathbb{C}^n \otimes \mathrm{triv}) \oplus (V_{(2,1)} \otimes \mathrm{std}) \oplus (\bigwedge^3 \mathbb{C}^n \otimes \mathrm{sgn})$.

Generalisation

Generalisations? Yes.

	Schur-Weyl Duality	Generalisation
V: vector space	$(\mathbb{C}^n)^{\otimes d}$ – finite-dimensional over $\mathbb{C}$	Infinite dimensional over $\mathbb{C}$
G: group	$\mathrm{GL}_n(\mathbb{C}) \times S_d$	reductive linear algebraic group
ρ: representation	irreducible rep of S_d, irreducible polynomial rep of $\mathrm{GL}_n(\mathbb{C})$	locally regular representations

Howe-Duality: Set-up

Let F be a non-archimedean local field. Let $(W, \langle\ ,\ \rangle)$ be a **symplectic vector space** over F of dimension $2n$, i.e., $\langle\ ,\ \rangle$ is a non-degenerate alternating bilinear form. Let $\mathrm{Sp}(W) := \{g \in \mathrm{GL}(W) : \langle gx, gy\rangle = \langle x, y\rangle\}$ be the **symplectic group** over W.

The **Heisenberg group** $H(W)$ is defined to be:

$$H(W) := W \oplus F = \{(w,t) : w \in W, t \in F\}$$

with the law of multiplication

$$(w_1, t_1) \cdot (w_2, t_2) = \left(w_1 + w_2, t_1 + t_2 + \frac{1}{2} \langle w_1, w_2 \rangle \right).$$

Facts:

1. There is an exact sequence $0 \to F \to H(W) \to W \to 0$.
2. The commutator subgroup is equal to the centre which is F, i.e., $[H(W), H(W)] = Z(H(W)) = F$.

Observe that $\mathrm{Sp}(W)$ acts on $H(W)$ by $g \cdot (w,t) = (g \cdot w, t)$.

Stone-von Neumann Theorem:

Up to isomorphism, there exists a *unique* smooth irreducible representation (ρ_ψ, S) of $H(W)$, with central character ψ, i.e., such that $\rho_\psi((0,t)) = \psi(t) \cdot \mathrm{id}_S$.

Since the action of any $g \in \mathrm{Sp}(W)$ is trivial on the center of $H(W)$, the twisted representation (ρ_ψ^g, S) given by $\rho_\psi^g(h) = \rho_\psi(h \cdot g)$ is isomorphic to (ρ_ψ, S) .

Oscillator representation

In particular, for each $g \in \mathrm{Sp}(W)$, there is an intertwining map $M(g) : S \to S$ such that

$$\rho_\psi(gh)M(g) = M(g)\rho_\psi(h). \tag{1}$$

The map $M(g)$ is only unique up to a scalar in $\mathbb{C}^\times$. So, the map $g \mapsto M(g)$ defines a projective representation

$$\rho : \mathrm{Sp}(W) \to GL(S)/\mathbb{C}^\times.$$

We wish to "de-projectivise" it.

Fix a central character ψ. The **metaplectic group** is defined to be

$$\widetilde{\mathrm{Sp}(W)} = \mathrm{Mp}(W) := \{(g, M(g)) \in \mathrm{Sp}(W) \times \mathrm{GL}(S) : (1) \text{ holds } \}.$$

Then, we can consider the (categorical) fiber product:

$$\begin{CD}
\mathrm{Mp}(W) := \widetilde{\mathrm{Sp}(W)} @>{\omega}>> \mathrm{GL}(S) \\
@VVV @VVV \\
\mathrm{Sp}(W) @>>> \mathrm{GL}(S)/\mathbb{C}^\times
\end{CD}$$

ω is called the **oscillator representation**

Howe-Duality analogy to Schur-Weyl Duality

	Schur-Weyl Duality	Howe Duality
V: vector space	$(\mathbb{C}^n)^{\otimes d}$ – finite-dimensional over $\mathbb{C}$	$S := \mathcal{C}_c^\infty(V)$, locally constant, compactly supported functions on V
G: group	$\mathrm{GL}_n(\mathbb{C}) \times S_d$	??
ρ: representation	irreducible rep of S_d, irreducible polynomial rep of $\mathrm{GL}_n(\mathbb{C})$	??

A *reductive dual pair* (G, G') in $\mathrm{Sp}(W)$ is a pair of subgroups G, G' in $\mathrm{Sp}(W)$ such that G and G' are (reductive groups and)

$$\mathrm{Cent}_{\mathrm{Sp}(W)}(G) = G' \quad \text{and} \quad \mathrm{Cent}_{\mathrm{Sp}(W)}(G') = G.$$

Examples:

- $(\mathrm{Sp}(W), \{\pm 1_W\})$ (Type I).
- $T \subset \mathrm{Sp}(W)$ which acts diagonally with respect to the standard basis. Then $\mathrm{Cent}_{\mathrm{Sp}(W)}(T) = T$, then (T, T) is a reductive dual pair (Type II).
- Let V be a symplectic vector space and U be an orthogonal vector space. $W = V \otimes U$ is a symplectic vector space and there are maps $j_1 : \mathrm{GL}(V) \to GL(W)$ and $j_2 : \mathrm{GL}(U) \to \mathrm{GL}(W)$. Then $(j_1(\mathrm{Sp}(V)), j_2(O(U)))$ is a reductive dual pair (Type I).

Howe-Duality analogy to Schur-Weyl Duality

	Schur-Weyl Duality	Howe Duality
V: vector space	$(\mathbb{C}^n)^{\otimes d}$ – finite-dimensional over $\mathbb{C}$	$S := \mathcal{C}_c^\infty(V)$, locally constant, compactly supported functions on V
G: group	$\mathrm{GL}_n(\mathbb{C}) \times S_d$	G, G', reductive dual pair
ρ: representation	irreducible rep of S_d, irreducible polynomial rep of $\mathrm{GL}_n(\mathbb{C})$	irreducible admissible representations

Howe Duality (kind of):

The representations of G and G' that occur in the oscillator representation correspond to each other bijectively. We have

$$\omega = \bigoplus_{\pi} \pi \otimes \pi',$$

where π and π' are irreducible admissible representations of G and G' respectively.

Howe-Duality: More precisely

Lemma (Moeglin-Vigneras-Waldspurger, 1987)

Let G_1, G_2 be two locally profinite groups. Let (π_1, V_1) be an irreducible admissible representation of G_1, (π, V) a smooth representation of $G_1 \times G_2$. Suppose that $\cap_f \mathit{ker}(f) = 0$ where f runs through $\mathit{Hom}_{G_1}(V, V_1)$. Then there exists a smooth representation (π_2, V_2) of G_2 unique up to isomorphism such that $\pi = \pi_1 \otimes \pi_2$.

Let G, G' be our dual pair. Now, pick (π, V_π) an irreducible admissible representation of $\widetilde{G}$. Define:

$$S(\pi) := \text{maximal quotient on } S \text{ on which } \widetilde{G} \text{ acts as a mutiple of } \pi$$

$S(\pi)$ is a $\widetilde{G}'$ and also $\widetilde{G} \times \widetilde{G}'$-module. Lemma $\Rightarrow S(\pi) = \pi \otimes \Theta(\pi)$.

Howe-Duality: More precisely

Howe Duality guarantees:

- $\Theta(\pi)$ is either 0 or is admissible and has finite length.
- If it is not zero, it contains a unique irreducible subquotient, which we call the "small theta lift," denoted $\theta(\pi)$. This is the partner representation for π.
- The map $\pi \mapsto \theta(\pi)$ is a one-to-one correspondence.

Howe-Duality: Applications

- Proved in all cases over local fields by Howe (1989), Waldspurger (1990), Minguez (2008), Gan and Takeda (2016) and Gan and Sun (2017).
- There is a global correspondence, where the oscillator representation is defined over adeles, and theta correspondences relate automorphic representations.
- Construction of counterexamples to the generalized Ramanujan–Petersson conjecture.