Lie algebras, root space decompositions and Dynkin diagrams

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These notes are from the second lecture I gave at a study group organised at King's College London on Bruhat – Tits theory.

Contents

Last time, we introduced Lie algebras, the lie bracket, λ-eigenspaces of lie algebras, root spaces, and the representation theory of finite Lie algebras (in particular, in terms of the representation theory of $\mathfrak{sl}_2$).

1 Abstract Root systems

Let $(E, (\cdot,\cdot))$ be a Euclidean space. Given $\alpha \in E$, define

$$\check{\alpha} : E \longrightarrow \mathbb{R}$$

given by

$$\check{\alpha} = \frac{2(\alpha, \lambda)}{\alpha, \alpha}.$$

(This looks very reminiscent of twice the projection formula $proj_{\mathbf{b}}(\mathbf{a}) = \frac{(\mathbf{a} \cdot \mathbf{b})\mathbf{b}}{(\mathbf{b} \cdot \mathbf{b})}$, and with a few diagrams ahead, we shall see it actually is.)

Notation. For $\mu \in E$, $\lambda \in E^*$, we define $\langle \mu, \lambda \rangle = \lambda(\mu)$, so, for example $\langle \alpha, \check{\beta} \rangle = \check{\beta}(\alpha)$.

Definition. A finite subset $\Phi \subset E$ is a *root system* if

- $0 \notin \Phi$, and Φ spans E
- If $\alpha, \beta \in \Phi$, then $\check{\beta}(\alpha) \in \mathbb{Z}$
- Define $w_\alpha : E \to E$ by $w_\alpha(\lambda) = \lambda - \check{\alpha}(\lambda)\alpha$. Then, if $\alpha \in \Phi$, $w_\alpha(\Phi) = \Phi$.
- If $\alpha, c\alpha \in \Phi$, for c a constant, then we must have $c = \pm 1$.

Example 1.1. Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{t} \in \mathfrak{g}$ a Cartan subalgebra. Let Φ be the set of root associated to $\mathfrak{t}$. Then, Φ forms a root system in $\operatorname{span}_{\mathbb{R}}(\Phi)$.

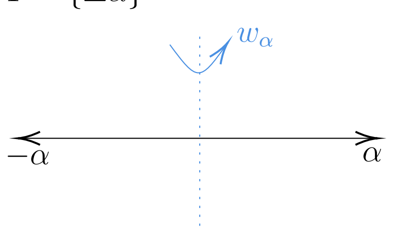
Definition. The dimension of E is called the *rank* of the root system.

Definition. If $\alpha \in \Phi$, then α is called a *co-root*. $\check{\Phi} := \{\check{\alpha} : \alpha \in \Phi\}$ is called the *dual root system* to Φ.

Definition. If (Φ, E) and (Φ', E') are root systems, then an *isomorphism* is a linear isomorphism $\rho : E \to E'$ such that

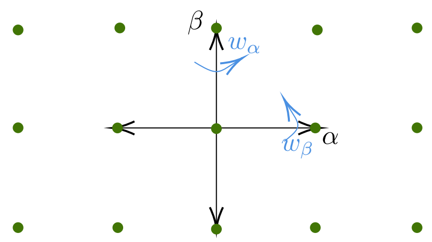
1. $\rho(\Phi) = \Phi'$
2. $\langle \rho(\alpha), \check{\rho}(\beta) \rangle = \langle \alpha, \check{\beta} \rangle$ for all $\alpha, \beta \in \Phi$.

Example 1.2. 1. **Dimension 1:** A_1 : (isomorphic to $\mathfrak{sl}_2$)
$\Phi = \{\pm\alpha\}$

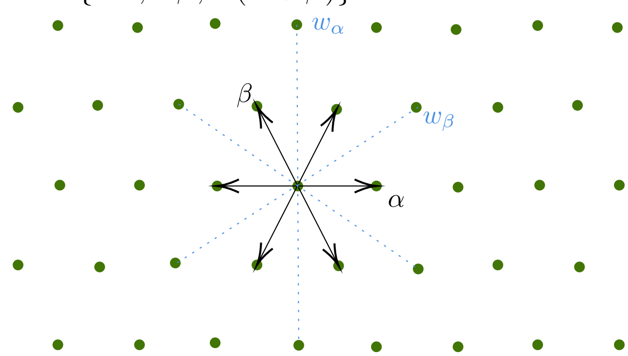


2. **Dimension 2**

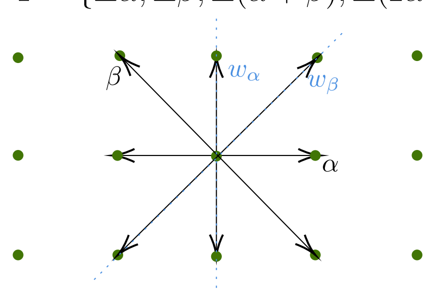
- $A_1 \times A_1$: (isomorphic to $\mathfrak{sl}_2 \oplus \mathfrak{sl}_2$)
  $\Phi = \{\pm\alpha, \pm\beta\}$



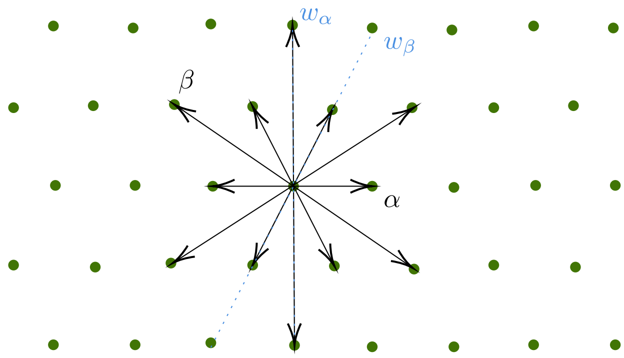
- A_2: (isomorphic to $\mathfrak{sl}_3$)
  $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$



- B_2: (isomorphic to $\mathfrak{sp}_4 \cong \mathfrak{so}_5$)
  $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta)\}$



- G_2 :
  $\Phi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta), \pm(3\alpha+\beta), \pm(3\alpha+2\beta)\}$



Definition. The *Weyl group* of (Φ, E) is the subgroup W of $GL(E)$ generated by $\{w_\alpha : \alpha \in \Phi\}$.

- $A_1 : W \cong C_2$
- $A_1 \times A_1 : W \cong V_4 \cong C_2 \times C_2$
- $A_2 : W \cong D_6 \cong S_3$
- $B_2 : W \cong D_8$
- $G_2 : W \cong D_{12}$

Fact 1.3. *If $(\Phi_1, E_1), (\Phi_2, E_2)$ are root systems, then $\Phi_1 \sqcup \Phi_2, E_1 \oplus E_2)$ is a root system.*

Lemma 1.4. *If Φ is a root system and $\alpha, \beta \in \Phi$, and $\alpha \neq \pm\beta$, then*

$$\langle \alpha, \check{\beta} \rangle \langle \beta, \check{\alpha} \rangle \in \{0, 1, 2, 3\}.$$

Proof. We use the fact that $(\alpha, \beta) = \sqrt{(\alpha, \alpha)}\sqrt{(\beta, \beta)} \cos\theta$, where θ is the angle between α and β. Then,

$$\langle \alpha, \check{\beta} \rangle \langle \beta, \check{\alpha} \rangle = \frac{4(\alpha, \beta)^2}{(\alpha, \alpha)(\beta, \beta)} = 4\cos^2\theta \in \mathbb{Z}$$

Since, $\cos^2\theta \in [0, 1]$, it must mean that $\cos^2\theta \in \{0, 1/4, 1/2, 3/4, 1\}$. But, as $\alpha \neq \pm\beta$, $\cos^2\theta \neq 1$, the result follows. □

Corollary 1.5. *If Φ is a root system and α, β are roots, then $\langle \alpha, \beta \rangle \in \{0, \pm 1, \pm 2 \pm 3\}$.*

2 Weyl Chambers

Notation. For (Φ, E) a roots system, for $\alpha \in \Phi$, we define

$$H_\alpha = \{\lambda \in E : \langle \lambda, \check{\alpha} \rangle = 0\}$$

to be the half plane corresponding to α.

Definition. The connected components of $E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$ are called *Weyl chamber*.

Definition. A subset $\Delta = \{\alpha_1, \ldots, \alpha_l\} \subset \Phi$ is called a *root basis* if

1. Δ forms a basis for E
2. $\alpha = \sum_{i=1}^{l} c_i \alpha_i \in \Phi$ such that $c_i \in \mathbb{Z}_{\geq 0}$ for all i or $c_i \in \mathbb{Z}_{\leq 0}$ for i.

Remark. The fact that such a basis exists is not obvious. For a proof, see Humphreys 10.1.

Example 2.1. The α, β drawn in the Example 1.2 are examples of root bases for each of the respective root systems.

Definition. If $\Delta = \{\alpha_1, \ldots, \alpha_2\}$ is a root basis, then α_i are called *simple roots*.

If $\alpha = \sum_{i=1}^{l} c_i \alpha_i$ with $c_i \geq 0$, it is called a *positive root*. If $\alpha = \sum_{i=1}^{l} c_i \alpha_i$ with $c_i \leq 0$, it is called a *negative root*.

We denote Φ^+ as the set of all positive roots and Φ^- as the set of all negative roots.

Remark. Δ defines a partial ordering on E, by saying $\mu \prec \lambda \iff \lambda - \mu$ is a sum of positive roots or $\lambda = \mu$.

Definition. We say $\gamma \in E$ is *regular* if $\gamma \in E \setminus \bigcup_{\alpha \in \Phi} H_\alpha$.

Notation. Choose $\gamma \in E \setminus \bigcup_\alpha H_\alpha$ (i.e., γ is in a Weyl chamber). Define

$$\Phi_\gamma^+ := \{\alpha \in \Phi : \langle \gamma, \check{\alpha} \rangle > 0\}$$

and

$$\Phi_\gamma^- := -\Phi_\gamma^+.$$

Then, $\Phi = \Phi_\gamma^+ \sqcup \Phi_\gamma^-$.

Further, define $\Delta_\gamma := \{\alpha \in \Phi_\gamma^+ \mid \alpha \neq \beta_1 + \beta_2, \text{ for all } \beta_1, \beta_2 \in \Phi_\gamma^+\}$.

We now state a few facts about root bases, proofs of which can be found in Humphreys.

Fact 2.2. *If Δ is a base, then $w(\Delta)$ is a base.*

Fact 2.3. *Δ_γ is a root basis.*

Fact 2.4. *Every root basis is of the form Δ_γ for some $\gamma \in E \setminus \bigcup_\alpha H_\alpha$.*

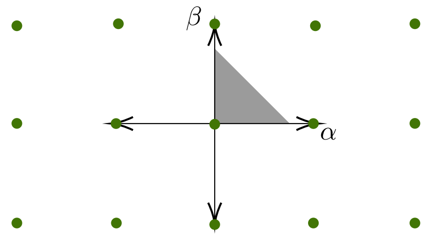
Fact 2.5. *There is a bijection between*

$$\{\textit{Weyl chambers}\} \longleftrightarrow \{\textit{root bases}\}.$$

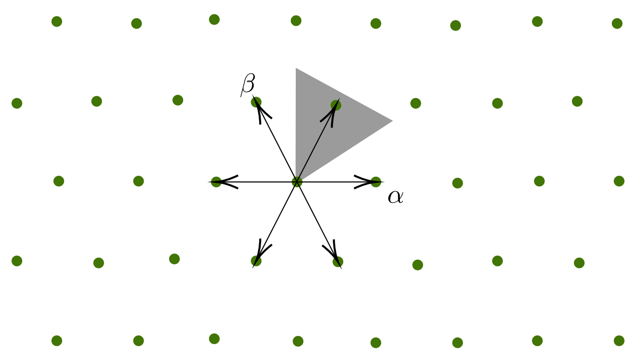
Definition 2.6. Given a root basis Δ_γ, the *fundamental Weyl chamber* is the Weyl chamber containing γ

Example 2.7. The shaded regions in the following diagrams show the fundamental Weyl chambers with respect to α, β.

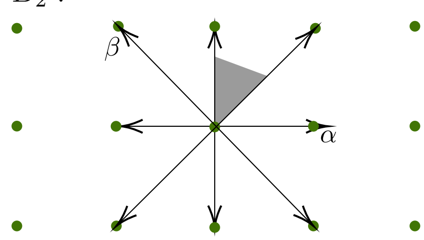
- $A_1 \times A_1$:



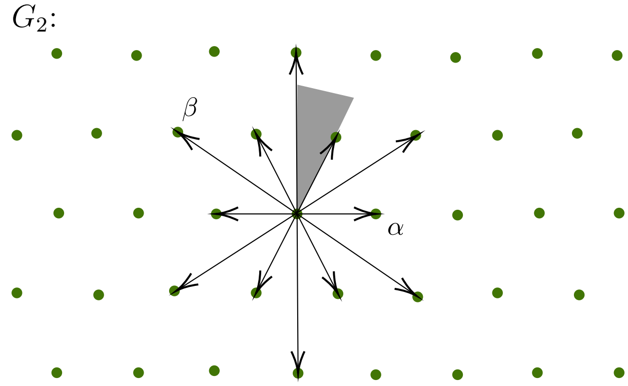
- A_2:



- B_2 :



- G_2:



Definition. When $w \in W$ can be written as a product of simple reflections $w = w_1 \dots w_t$, where each $w_i = w_{\alpha_j}$ for some j, $\alpha_i \in \Delta$, and t is minimal, we call the expression *reduced*. We write $l(w) = t$. This is length of w relative to Δ.

Fact 2.8. $l(w) =\mid w(\Phi^+) \cap \Phi^- \mid$.

Fact 2.9. *There exists a unique $w_0 \in W$ with maximal length.*

Fact 2.10. *Given $\alpha \in \Phi$, $w \in W$, $w_{w(\alpha)} = w^{-1} w_\alpha w$.*

Theorem 2.11.

1. *The Weyl group acts simply transitively on the set of root bases and the set of Weyl chambers.*
2. *Given a root basis Δ and $\alpha \in \Phi$, there is a $w \in W$ with $w(\alpha) \in \Delta$.*
3. *If $\Delta = \{\alpha_1, \dots, \alpha_l\}$ is a root basis, then W is generated by $\{w_\alpha \mid \alpha \in \Phi\}$.*

Before we prove the theorem, we prove the following lemmas:

Lemma 2.12. *Let α be simple. Then w_α permutes the positive roots other than α.*

Proof. Let $\beta \in \Phi^+ \backslash \{\alpha\}$. Then, we can express $\beta = \sum_{\gamma \in \Delta} k_\gamma \gamma$, where $k_\gamma \in \mathbb{Z}_{\geq 0}$, and as $\beta \neq 0$, at least one k_γ is non-zero and positive. Note that α lies in Δ because α is simple. So, we can write

$$\begin{aligned} w_\alpha(\beta) &= \beta - \check{\alpha}(\beta)\alpha \\ &= \sum_{\gamma \neq \alpha} k_\gamma \gamma + d\alpha \end{aligned}$$

for some constant d. But, as $\beta \in \Phi^+ \backslash \{\alpha\}$, it must have all positive coefficients, which forces d to also be non-negative.

Further, $w_\alpha(\beta) \neq \alpha$ as α has image $-\alpha$. □

Corollary 2.13. *Set $\delta = (1/2) \sum_{\beta \succ 0} \beta$. Then, $w_\alpha(\delta) = \delta - \alpha$ for all $\alpha \in \Delta$.*

Lemma 2.14. *Let $w = w_1 \dots w_t$ be an expression for $w \in W$ in terms of reflections corresponding to simple roots with t minimal. Then, $w(\alpha_t) \prec 0$.*

Proof. See Humphreys 10.2 Lemma C Corollary. □

Proof of 2.11. Define $W' = \langle w_\alpha : \alpha \in \Delta \rangle$. We will first show that the action by the group of simple reflections W' is transitive and then show $W = W'$.

1. Pick $\gamma \in E \backslash \bigcup_{\alpha \in \Phi} H_\alpha$ and let $\delta = (1/2) \sum_{\alpha \succ 0} \alpha$. Choose $w \in W'$ such that $(w(\gamma), \delta)$ is as big as possible. If α is simple, $w_\alpha w \in W'$ (by Lemma 2.12). So,

$$\begin{aligned} (w(\gamma), \delta) &\geq (w_\alpha w(\gamma), \delta) \\ &= (w(\gamma), w_\alpha(\delta)) \\ &= (w(\gamma), \delta - \alpha) \\ &= (w(\gamma), \delta) - (w(\gamma), \alpha) && \text{(Corollary 2.13)} \\ \Rightarrow (w(\gamma), \alpha) &\geq 0 \quad \forall \alpha \in \Delta \end{aligned}$$

Since γ is regular, we cannot have $(w(\gamma), \alpha) = 0$ for any α, otherwise γ would be orthogonal to $w^{-1}(\alpha)$, so all the inequalities are strict. This implies that $w(\gamma)$ lies in the fundamental Weyl chamber $C(\Delta)$, and so w sends $C(\Delta_\gamma)$ to $C(\Delta)$. Since W' permutes the Weyl chambers, it also permutes the bases of Φ by Fact 2.5. We will show simple transitivity at the end (okay, that's a lie).

2. It suffices to show that each root belongs to at least one base. Since the only root proportional to α is $\pm\alpha$, the hyperplane H_β ($\beta \neq \pm\alpha$) are distinct from H_α, so there exists a $\gamma \in H_\alpha$, $\gamma \notin H_\beta$. Chose γ' close enough to γ such that $(\gamma', \alpha) = \varepsilon > 0$, while $|(\gamma, \beta)| > \varepsilon$ for every $\beta \neq \pm\alpha$ (we can do this because of the way we have chosen γ). Then, it follows that $\alpha \in \Delta(\gamma')$.

3. To prove $W = W'$, it is enough to show that the reflection w_α, $\alpha \in \Phi$ is in W'. Using part 2. above, find $w \in W'$ such that $\beta = w(\alpha) \in \Delta$. Then, $w_\beta = w_{w_\alpha} = w w_\alpha w^{-1}$ (using Fact 2.10). So, $w_\alpha = w^{-1} w_\beta w \in W'$.

To see the proof of simple transitivity, refer Humphreys 10.3 Theorem. □

3 Dynkin Diagrams

Let (Φ, E) be a root system and a root basis $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. Let W be a Weyl group.

Definition. The *Cartan matrix* of Φ is the $l \times l$ matrix given by $C = [\langle \alpha_i, \check{\alpha}_j \rangle_{1 \leq i,j \leq l}]$.

Remark. This is independent of the choice of root basis since if Δ' is another root basis, then there exists w such that $w(\Delta) = \Delta'$ and w preserves $\langle \cdot, \cdot \rangle$ up to permutation.

Example 3.1.

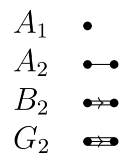
- $A_1 : C = (2)$
- $A_1 \times A_1 : C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$
- $A_2 : C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$
- $B_2 : C = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$
- $G_2 : C = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

Definition. The Dykin Diagram of Φ has

1. vertices $\leftrightarrow \Delta$.
2. i^{th} and j^{th} vertices are connected by $\langle \alpha_i, \check{\alpha}_j \rangle \langle \alpha_j, \check{\alpha}_i \rangle$ edges.

3. For multiple edges, an arrow points to the shorter root.

Example 3.2.

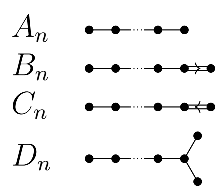


Remark. • Maximum number of edges between 2 vertices in a Dynkin Diagram is 3, and a root system is called *simply laced* if and only if its Dynkin Diagram has no multiple edges.

• Φ is irreducible $\iff$ Dynkin diagram is (simply) connected.

3.1 Classification of root systems

3.2 Classical root systems (with rank l)



Type	W	$\mathfrak{g}$	dimension
A_l	S_{l+1}	$\mathfrak{sl}_{l+1}(\mathbb{C})$	l^2-1
B_l	$S_l \ltimes (C_2)^l$	$\mathfrak{so}_{2l+1}(\mathbb{C})$	$2l^2+l$
C_l	$S_l \ltimes (C_2)^l$	$\mathfrak{sp}_{2l}(\mathbb{C})$	$2l^2+l$
D_l	$S_l \ltimes (C_2)^{l-1}$	$\mathfrak{so}_{2l}(\mathbb{C})$	$2l^2-l$

3.3 Exceptional root systems